

EXTERIOR BALLISTICS

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1935

A textbook of those elements of exterior ballistics which
are the fundamentals of naval gunnery, written for
the use of midshipmen at the United States
Naval Academy

By

ERNEST E. HERRMANN

Lieutenant Commander, United States Navy

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PREFACE

Exterior Ballistics, 1935 has been prepared to serve principally as a textbook for the instruction of midshipmen at the United States Naval Academy, and its scope is confined to material which is appropriate for the training of naval officers in the problems of naval gunnery. Although a number of chapters deal with fundamentals that are general in their application to the problems of exterior ballistics, complete development of practical methods is confined to those which are, at the time of this writing, considered standard in the United States Navy. The notation, likewise, is in accordance with usage in the United States Navy. These features limit the usefulness of this volume as a general reference book, and it is not intended that it should be regarded as such.

In the preparation of this volume, close attention has been given to the limitations imposed upon it by its special purpose. These limitations apply both to the time allotted to the subject in the curriculum of the United States Naval Academy and to the mathematical equipment of the students for whom the book is designed. In view of the limited time available to midshipmen for the preparation of lessons, an effort has been made to include in the text thorough explanations of features which, in the past, have proved troublesome and have resulted in a drain on recitation periods. In general, the formulas given in the text are preceded by complete derivations, in order that the student may have full proof of the methods employed without being obliged to consult reference books. In a few cases, however, this would lead to digressions of greater length than justified by the ends to be gained; in such cases reference is made to sources in which details omitted from the text may be found.

Although its principal purpose is to serve as a textbook for undergraduate study, this volume has been designed to serve also as a reference book for those who desire to extend their studies of exterior ballistics beyond the scope of the undergraduate course. Treatments of certain special features are outlined in footnotes for this purpose. Also, reference is made to numerous sources which are appropriate for collateral study. It is not intended, however, that consultation of these sources should constitute a part of the undergraduate course.

The various tables that are required in connection with this textbook are published as a separate volume, entitled *Range and Ballistic Tables, 1935*. The contents of the latter volume are stated in its introductory pages.

Exterior Ballistics, 1935 is the successor to *Exterior Ballistics, 1930* and *Exterior Ballistics, 1926*, which were prepared by the author of the present volume. Although much of the material appearing in the 1926 and 1930 editions has been retained in the present volume, the latter has been completely rewritten and is essentially a new textbook. The 1926 and 1930 editions covered in detail the computation of range tables by both the Ingalls-Siacci Method and the numerical-integration method. Siacci's Method, with its various outgrowths, was the primary method for the computation of the U.S. Navy range tables from about 1890 until about 1920; since about 1920, Siacci's Method has been restricted to the computation of range-table values for trajectories having angles of departure not exceeding 15° , and the numerical-integration method has been used in connection with trajectories having angles of departure greater than 15° . It is anticipated that the numerical-integration method will, in the near future, replace Siacci's Method altogether. Accordingly, the latter is treated in the present volume as an obso-

lescent method, and the practical features involved in the computation of modern range tables are dealt with primarily according to the numerical-integration method. The scope of the book has been extended to include practical methods for dealing with the effects of variations in the assumed standard air structure and of non-uniform winds, as determined by aloft soundings, and with the effects of the earth's rotation. Advantage has been taken of the extensive revision outlined above to modernize the subject matter throughout and to condense and simplify the presentation.

A complete statement of those to whom acknowledgment for contributions to this textbook is due would involve the naming of individuals who, during a period of almost fifty years, have been the authors of and contributors to the United States Naval Academy textbook on exterior ballistics. The first of these textbooks was *Exterior Ballistics, 1887*, by Lieutenant J. F. Meigs, U.S. Navy, and Lieutenant R. R. Ingersoll, U.S. Navy, and it was succeeded, in turn, by the following: *Exterior Ballistics, 1893* and *Exterior Ballistics, 1901*, by Lieutenant Commander R. R. Ingersoll, U.S. Navy; *Exterior Ballistics, 1904* and *Exterior Ballistics, 1906*, by Professor P. R. Alger, U.S. Navy; *The Groundwork of Practical Naval Gunnery, or Exterior Ballistics, 1915*, by Professor P. R. Alger, U.S. Navy; and *Exterior Ballistics, 1926* and *Exterior Ballistics, 1930*, by the author of the present volume. Although the 1915 book appeared as a revision of Professor Alger's previous works, it represented, in fact, a very considerable extension beyond the scope of the latter, and was actually prepared by Captain L. H. Chandler, U.S. Navy. Captain Chandler's book, in many respects, was used as a model by the present author. Acknowledgments are due, and are gratefully made, to all of the above mentioned sources and to the contributors mentioned by their respective authors, as well as to numerous other sources to which reference is made at appropriate points in the text.

Exterior Ballistics, 1926 was prepared under the immediate direction of Captain Walter S. Anderson, U.S. Navy, then Head of Department of Ordnance and Gunnery, U.S. Naval Academy. The author owes the accomplishment of his tasks in connection with the 1926, 1930, and 1935 editions of this textbook in large measure to the continued confidence and encouragement of Captain Anderson.

The author is indebted also to the following for immediate assistance in the preparation of the present volume: Captain G. L. Schuyler, U.S. Navy, Bureau of Ordnance, U.S. Navy Department, for helpful criticism of the 1926 and 1930 editions; Mr. J. W. Webb, Bureau of Ordnance, U.S. Navy Department, and Mr. S. Feltman, Technical Staff, Ordnance Department, U.S. Army, for advice on certain technical details. The author has sought to embody in this work the suggestions of all instructors in the Department of Ordnance and Gunnery, U.S. Naval Academy, principally as to the manner of presenting features which have in the past given difficulty to the student, and is, to this extent, indebted to all officers who have served with him in this Department.

ERNEST E. HERRMANN

Lieutenant Commander, U.S. Navy,
Department of Ordnance and Gunnery,
U.S. Naval Academy.

1 September 1935.

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CHAPTER 1

INTRODUCTION

101. The science of exterior ballistics has for centuries engaged some of the world's best mathematical genius. A treatise on the flight of projectiles was published by Nicholas Tartaglia in 1537, and this may be regarded as the beginning of what has since become a very extensive literature on this subject. Galileo contributed to the investigations of the trajectory in vacuum, and Newton considered the effects of air resistance on the trajectory. As early as 1753, Euler gave an approximate solution of the trajectory in air that is of particular interest because of its close approach to the only recently adopted methods of our own day. These early treatments of the problem, and the almost innumerable ones that have followed them and have lead to the methods now in use, form a classical background well worth study by students so inclined.

The aim of this textbook is to treat the science of exterior ballistics from the viewpoint of its practical applications to the problems of modern naval gunnery. Occasionally it is useful, in this connection, to trace historically the development of some features, in order that the line of progress may be indicated. The time available for the study of this course at the Naval Academy is not sufficient to permit extensive consideration of alternative methods. The methods explained in this text are limited, in general, to those employed in the U.S. Navy. However, reference is made frequently to sources in which unlimited treatments may be found.

102. The ultimate purpose of this textbook is to teach naval officers how to direct the fire from the guns of a ship in the most effective manner. In order to accomplish this purpose, the problem is treated in three stages, as follows:

- A. First to be considered is the problem of solving the trajectory in air. In this stage we find occasion to deal extensively with theory. Two distinct problems are met in this portion of our study, namely, (1) investigation and measurement of the forces which operate on a projectile in flight, and (2) the establishment of methods by means of which all of these forces may be accounted for in a solution of the trajectory, with a degree of accuracy commensurate with the requirements of practice. This stage embraces also the development of tables by means of which the solution of a trajectory may be facilitated; such tables are called *ballistic tables*. This stage is not confined to any particular branch of gunnery, but deals with fundamental applications of the theories of exterior ballistics to the solution of the trajectory.
- B. Next to be considered is the problem of reducing the general information contained in ballistic tables to the specific information required for any given gun for all circumstances attending its use. This problem is handled by preparing for each gun a *range table* in which all data required for the control of that gun are tabulated in convenient form. In this stage we shall limit ourselves to the problems involved in constructing range tables in the form in which the latter are used in the U.S. Navy. The range

tables used by other services may differ materially, in form, from our own.

- C. Finally we shall deal with the problem of using range tables for the determination of firing data, and with the application of the laws of errors to the analysis of results. In this final stage we shall deal with the immediate, practical applications of the science of exterior ballistics to the problems involved in the control of gunfire from ships.

In actual practice it is ordinarily not necessary to deal specifically with matters outlined in the first two stages; each ship is provided with range tables for its guns, and the ordinary shipboard problem is one of using these tables. It is to be expected, however, that a more intelligent use of range tables will result from prior consideration of the problems entering into their construction. Also, the possibility of having to use a ship's guns in a manner not provided for by available range tables should not be overlooked. Such instances have occurred in the past. Complete mastery of a ship's armament, under any circumstances that may arise, demands the ability to prepare firing data from the ground up, if necessary.

103. The methods of exterior ballistics may be called semi-empirical. No exact solution of a trajectory in air exists, and it is not probable that one will ever be devised. The factors governing the flight of a projectile are numerous and, even under the best of conditions, some of these factors are of an uncertain nature. We refer to accurate solutions and to approximate solutions, but these terms are only relative. The accuracy of any solution is limited ultimately by the precision with which the physical values entering into it can be measured; some of these will probably always remain indeterminable to a certain degree. In vacuum, the trajectory is a true parabola and its exact solution offers no difficulty. The presence of an air medium introduces many complications; some of these can be handled effectively only by deducing from the results of actual firing empirical constants which may be used in connection with mathematical solutions.

104. Numerical exercises in connection with the first two stages outlined in article 102 (embracing Chapters 2 to 11) are introduced in order to ensure familiarity with the sources of data for making trajectory solutions, as well as to promote that degree of understanding of processes which can come only from actually using them. The problem of assigning appropriate standards of computational accuracy in connection with such exercises has always been a troublesome one. But despite the various considerations that may be advanced in support of greater latitude in the matter, adherence to uniform methods and standards is required in the solutions submitted by midshipmen in order that errors in the solutions may be traced readily. In the strictly practical exercises of the third stage outlined in article 102, the considerations discussed above apply equally well, and, in addition, correctness of the result is of itself an important end, for these exercises are to be regarded as training for similar work to be done in actual service. The real significance of accuracy in a strictly practical problem in gunnery often depends largely upon the immediate circumstances of the case involved. It is well known that in actual practice unaccountable errors often occur. But the fact that a target is often missed on the first salvo despite the best of efforts to determine the ballistic corrections accurately, does not render such efforts futile, any more than the fact that unknown ocean currents often carry a ship from her designed course renders accurate navigational methods futile. Actual experience is the only safe guide as to the degree of accuracy that is appropriate

under a given set of conditions. The midshipman, in working the exercises given in this book, is expected to adhere to the standards of accuracy that are used in the solutions illustrated in the text.

LETTERS OF THE GREEK ALPHABET USED AS SYMBOLS.

Letter. Pronunciation.	Letter. Pronunciation.	Letter. Pronunciation.
α Alpha.	θ Theta.	Σ or σ Sigma.
β Beta.	λ Lambda.	ϕ Phi.
γ Gamma.	μ Mu.	ψ Psi.
Δ or δ Delta.	π Pi.	Ω or ω Omega.
ϵ Epsilon.	ρ Rho.	

CHAPTER 2

DEFINITIONS AND INTRODUCTORY EXPLANATIONS. PRELIMINARY ASSUMPTIONS.

Symbols Introduced

X	Horizontal range.
X'	Inclined range.
x	Abscissa of any point in the trajectory.
x_s	Abscissa of the summit.
y	Ordinate of any point in the trajectory.
y_s	Ordinate of the summit, or maximum ordinate.
D	Drift.
p	Angle of position.
ϕ	Angle of departure.
ϕ'	Angle of elevation.
j	Angle of jump.
ω	Angle of fall.
θ	Angle of inclination at any point in the trajectory.
v	Remaining velocity at any point in the trajectory.
V	Initial velocity (Also I.V.).
v_w	Striking velocity.
v_s	Summital velocity.
v_h	Horizontal velocity at any point in the trajectory.
v_v	Vertical velocity at any point in the trajectory.
t	Time of flight to any point in the trajectory.
T	Time of flight to point of fall.
t_s	Time of flight to the summit.

201. Ballistics is the science of the motion of projectiles. It is divided into two branches, namely, interior ballistics and exterior ballistics. Interior ballistics is that branch of the science which treats of the motion of the projectile while in the gun and of the phenomena which cause and attend this motion. Exterior ballistics treats of the motion of the projectile after it leaves the gun. The subject of interior ballistics is covered in the current edition of *Naval Ordnance* and will not be dealt with further here.

The path described by a projectile in flight is called the *trajectory*. Although the term *trajectory* is sometimes also applied to the path of a projectile through a medium other than air (for example, we speak of the under-water trajectory in dealing with the projectile's path through the water after impact on the surface of the sea), the term is commonly confined to the path of the projectile from the muzzle of the gun to the first point of impact, and we shall deal with it here in that sense. The following definitions pertain to elements of the trajectory. All of the angles defined are to be considered as measured in the vertical plane.

202. The *line of position* is the straight line joining the gun and target (OP , Figure 1).

The *angle of position* (p) is the angle between the horizontal plane and the line of position; it is positive when the target is higher than the gun, and negative when the target is lower than the gun.

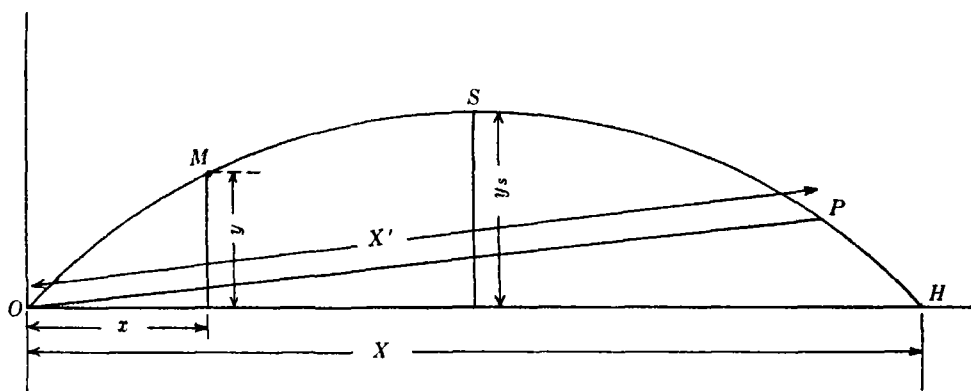
203. The *point of fall*, unless otherwise qualified, is the point at which the descending branch of the trajectory intersects the horizontal plane through the origin. For a target not in the latter plane, the point of fall may be taken as the point beyond the origin at which the trajectory intersects the line of position.

204. The distance in a straight line from the origin to a point of fall in the horizontal plane through the origin, is called the *horizontal range* (OH , Figure 1); it is denoted by the symbol X .

The distance in a straight line from the origin to a point of fall *not* in the horizontal plane through the origin, is called the *inclined range* (OP , Figure 1); it is denoted by the symbol X' .

As we shall ordinarily deal with the case of a point of fall in the horizontal plane through the origin, the term *range* will be used for convenience to denote horizontal range as defined above; the distinction provided for above will be made only when necessary.

The abscissa of any point in the trajectory (other than the point of fall in the horizontal plane through the origin) is denoted by the symbol x ; for the summit it is denoted by x_s .



X = Horizontal range
 X' = Inclined range
 x = Abscissa of point M

y = Ordinate of point M
 y_s = Maximum ordinate

FIGURE 1

205. The *summit*, or vertex, of the trajectory is the point at which the projectile ceases to ascend and commences to descend. If an actual trajectory should terminate in the ascending branch (as may occur in antiaircraft fire), the summit, in the sense here intended, will lie on an imaginary extension forward of the real trajectory. Similarly, if a gun should be fired downward so that the trajectory has only a descending branch, the summit will lie on an imaginary extension backward of the real trajectory.

The ordinate at the summit of the trajectory is called the *maximum ordinate*; it is denoted by the symbol y_s .

The ordinate at any other point in the trajectory is denoted by the symbol y .

206. The *line of departure* is the line in which the projectile is moving at the instant of projection; it is tangent to the trajectory at the origin.

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The *angle of departure* (ϕ) is the angle between the horizontal plane and the line of departure (BOH , Figure 2).

The *angle of elevation* (ϕ') is the angle between the line of position and the line of departure (*BOP*, Figure 2).

The *angle of jump* (j) is the angle described by the axis of the bore, under the shock of firing, during the interval from the ignition of the charge to the ejection of the projectile from the muzzle (BOA, Figure 2); it is positive when the muzzle of the gun jumps upward, and negative when the muzzle of the gun jumps downward. In well-designed, modern guns the angle of jump amounts only to a few minutes of arc and remains reasonably constant, for a given gun, for all angles of departure; it is generally positive, but not invariably so.

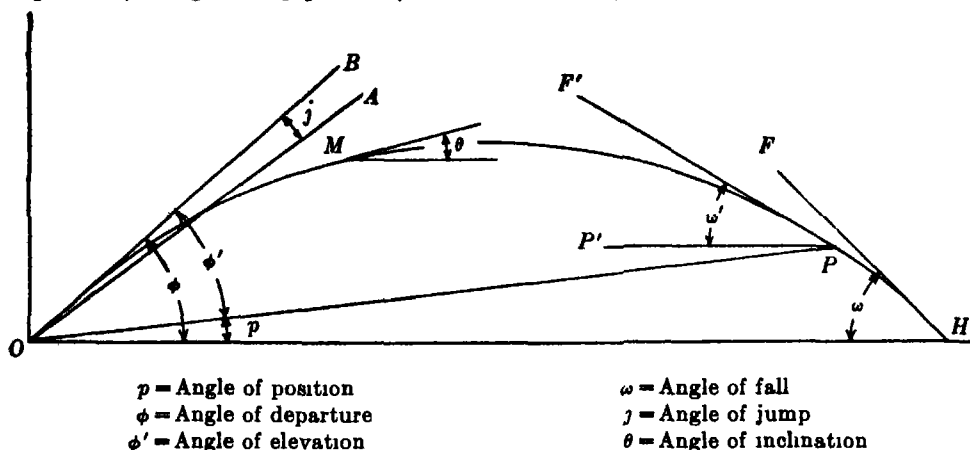


FIGURE 2

It is well to note here the relation between angle of departure and angle of elevation. When establishment of the true horizontal is impracticable, as is generally the case on board ship, some other plane of reference must be chosen. The line of sight to the target, which can be established by sighting on the target, affords such a plane. If the line of sight is not horizontal, the angle to be set with respect to the line of sight must be adjusted accordingly. The line of sight is evidently identical with the line of position as already defined above, and hence the angle of position measures the inclination between the line of sight and the horizontal. The same result will be obtained with the gun elevated the angle $\phi' = \phi - p$ above the line of sight as will be obtained with gun elevated the angle ϕ above the horizontal. The angle ϕ' , as here defined, is then the angle at which the gun-sight telescopes must be set with respect to the bore, in order to give the gun the desired angle of elevation above the line of position (or sight). Strictly speaking, the angle to be set on the sights, and hence the angle of elevation, differs from the angle at which projection will actually occur, by the amount of the angle of jump. Or, in other words, the angle of elevation, in the sense that it measures the elevation of the bore above the line of sight just before firing, is really $\phi' - j$. However, the angle of jump is sufficiently small and uniform to allow it to be handled as an inherent part of the angle of elevation, and no distinction need be made between the angle of elevation and the angle of projection. This will be further clarified in article 804.

207. The *angle of fall* (ω) is the angle between the horizontal plane and the tangent to the trajectory at the point of fall (FHO , Figure 2). It is to be noted that the angle of fall is always measured with respect to the horizontal plane.

even when the point of fall is not in the horizontal plane through the origin. Thus at the elevated target, P , in Figure 2, the angle of fall is $\omega' = F'PP'$.

The *angle of inclination* (θ) at any point in the trajectory is the angle between the horizontal plane and the tangent to the trajectory at the given point. The angle of departure (ϕ) and angle of fall (ω) are, of course, special values of the angle of inclination applying, respectively, to the origin and point of fall.

208. The *initial velocity* (V) is the velocity with which the projectile is supposed to leave the muzzle of the gun (the abbreviation I.V. is frequently used). The term *muzzle velocity* (abbreviated M.V.) is synonymous with initial velocity, but the latter term is generally used in exterior ballistics.

Since it is impracticable, for reasons that are readily apparent, to obtain a direct measure of the velocity of a projectile at the instant it leaves the muzzle, it is the practice to measure the velocity at as short a distance from the muzzle as practicable and to deduce therefrom the initial velocity. The process by which this is accomplished will be gone into later (arts. 806-808). It may be noted here, however, that the process employed assumes that the projectile has suffered normal retardation during its flight from the muzzle to the point of measurement. Actually, immediately after leaving the muzzle, the projectile is surrounded by a blast of gases moving more rapidly than the projectile, and experiments have shown that the projectile is accelerated during a travel of as much as fifty yards from the muzzle. Thus the value of initial velocity deduced in the manner just outlined is in fact a fictitious value; it is more accurately defined as the equivalent initial velocity, assuming projection into still air, that would produce the remaining velocity actually measured at a short distance from the muzzle.

209. The *remaining velocity* (v) at any point in the trajectory is the velocity at that point measured in the tangent to the trajectory at that point.

The *striking velocity* (v_s) is the remaining velocity at the point of fall.

The *summital velocity* (v_s) is the remaining velocity at the summit, or vertex, of the trajectory.

The *horizontal velocity* (v_h) at any point in the trajectory is the horizontal component of the remaining velocity at that point.

The *vertical velocity* (v_v) at any point in the trajectory is the vertical component of the remaining velocity at that point.

210. The *time of flight*, when denoted by the symbol T , is the total time that elapses in the flight of the projectile from the gun to the point of fall. The time of flight to any other point in the trajectory is denoted by the symbol t .

211. The *drift* (D) is the perpendicular distance of the point of fall from the vertical plane containing the line of departure (HH' , Figure 3).

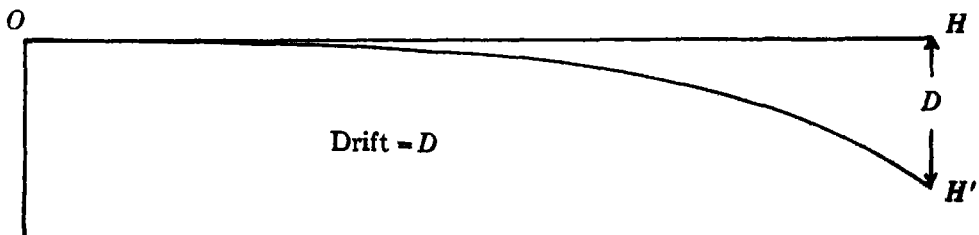


FIGURE 3

212. The units in which values of the various elements of the trajectory are expressed must be given careful consideration in numerical problems, since

current practice as to the use of units is by no means uniform. In the range tables of the U.S. Navy the following is standard practice:

- Units**
- (a) Velocities are expressed in foot-seconds.
 - (b) Horizontal distances are expressed in yards.
 - (c) Vertical distances are expressed in feet.
 - (d) Angles are expressed in sexagesimal units.

Exercises in this book ordinarily will develop solutions in these units.

It is to be noted, however, that many of the sources of ballistic data used by us have originated outside of our own service, and usually are in metric units. In these sources all distances are commonly expressed in meters, and velocities in meter-seconds. In order to avoid the multiplicity of symbols that would result from adopting separate symbols for the several elements to denote the different units in which they may be expressed, no distinction as to units will be attached to symbols. Due caution must be exercised, therefore, to state the units applicable whenever numerical values are involved.

Preliminary assumptions 213. Before proceeding with the problem of solving the trajectory the following preliminary assumptions will be made:

1. The dimensions of the gun are negligible in comparison with the trajectory; for convenience we shall assume the origin of the trajectory to be at the intersection of the bore with the trunnion axis of the gun, and the trunnion axis to be horizontal and located at the surface of the earth.
2. All positions will be referred to a system of rectangular axes, with the origin coinciding with the origin of the trajectory as defined above; and with the *X*-axis horizontal, perpendicular to the trunnion axis, and positive in the direction of fire; the *Y*-axis vertical and positive upward; and the *Z*-axis horizontal, perpendicular to the *X*-axis, and positive to the right with respect to the direction of fire. This results in the establishment of a datum plane tangent to the earth at the origin, and this system of coordinates in exterior ballistics is therefore referred to as the *tangent-plane system*. The fact that the surface of the earth departs from this tangent plane may be accounted for by very simple corrections whenever necessary, as will be brought out later.*
3. The force of gravity acts always in a direction perpendicular to the datum plane established above.*
4. The earth and its atmosphere and the gun itself all are motionless. These assumptions are made only for the purpose of obtaining a first approximation of the motion of the projectile. Later we shall take into account separately the effects of the earth's motion, and of wind and motion of the gun.
5. The axis of the projectile remains at all points coincident with the tangent to the trajectory. Under this assumption, since both the initial force imparted by the gun, and the force of gravity, are exerted in the vertical plane containing the line of departure, the force of air resistance is confined to this plane, and the trajectory is therefore a plane curve confined entirely to the vertical plane containing the line of departure. This assumption is made only while dealing with a first approximation of the motion of the

* In some works on exterior ballistics positions are referred to a curved system which may be thought of as the usual Cartesian grid bent to conform to the surface of the earth. This system has no material advantages over the tangent-plane system, and, as will appear later (article 813), the latter system actually is the one more applicable to problems in naval gunnery.

CHAPTER TWO

projectile. Later we shall take into account separately the effects of forces not satisfying this assumption.

The above assumptions are general in their application and serve principally to strip the problem of certain complications which can more readily be dealt with after the principal effects have been accounted for. Additional assumptions will be introduced at appropriate points later in the text.

EXERCISES

1. For the following angles of elevation and position, what are the corresponding angles of departure?

Problem	DATA				ANSWERS	
	Angle of elevation		Angle of position		Angle of departure	
1.....	2°	00'	+15°	00'	+17°	00'
2.....	3	00	+12	15	+15	15
3.....	3	00	-10	30	-7	30
4.....	2	00	-12	07	-10	07
5.....	3	00	+11	15	+14	15
6.....	5	00	+10	16	+15	16
7.....	4	00	-9	37	-5	37
8.....	6	00	-6	22	-0	22

2. A target is at a horizontal distance of 3000 yards from the gun, and is 750 feet higher than the gun above the water. Compute the angle of position.

Answer. $p = 4^{\circ} 45' 49''$.

3. A target is at a horizontal distance of 10,000 yards from the gun, and is on the water 1500 feet below the level of the gun, the latter being in a battery on a hill. Compute the angle of position.

Answer. $p = (-)2^{\circ} 51' 45''$.

4. A target is at a horizontal distance of 1924 yards from the gun, and is 1123 feet higher above the water than the gun. Find the angle of position and the distance in a straight line from the gun to the target in yards.

Answers. $p = 11^{\circ} 00' 35''$. $X' = 1960$ yards.

5. A target is at a horizontal distance of 1860 yards from the gun, and it is on the water 1238 feet below the level of the gun, the latter being in a battery on a hill. Find the angle of position and the distance in a straight line from the gun to the target in yards.

Answers. $p = (-)12^{\circ} 30' 33''$. $X' = 1905$ yards.

CHAPTER 3

THE EQUATION OF THE TRAJECTORY IN VACUUM. THE THEORY OF RIGIDITY OF THE TRAJECTORY.

301. Although the resistance of the atmosphere to the motion of projectiles traveling at high velocities is sufficiently great to render any calculations which neglect such resistance worthless for giving information for actual firing, certain similarities exist between the trajectory in vacuum and the trajectory in air; these similarities pertain rather to general characteristics of the trajectory than to any quantitative values of the elements. Since the solution of the trajectory in vacuum is very simple, while the solution under conditions of air resistance becomes very complex, it is desirable to make a preliminary study of the general characteristics of the trajectory from a study of the curve in vacuum. From this study it will also be possible to make some deductions that will have an important application to practical problems dealing with the trajectory in air.

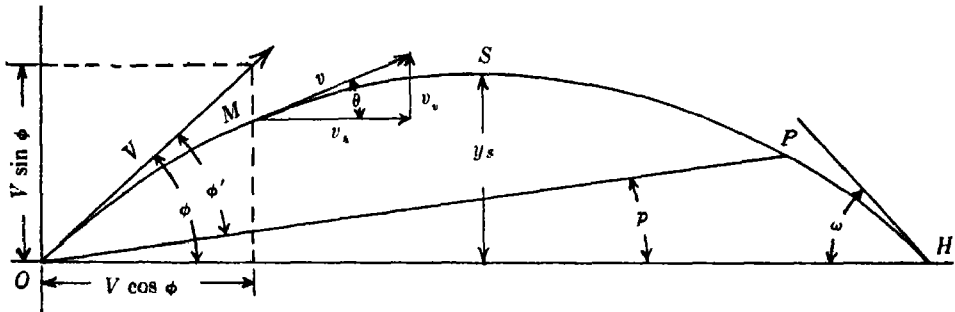


FIGURE 4

302. In Figure 4, the curve OSH represents the trajectory in vacuum, with origin at O and point of fall at H in the horizontal plane through O . The line marked V is the line of departure, its length representing the initial velocity V ; the horizontal and vertical components of the latter, $V \cos \phi$ and $V \sin \phi$, respectively, are also shown.

Since the force of gravity is exerted only in the vertical plane, the horizontal distance (x) covered by the projectile in any interval of time (t) is the horizontal component of the initial velocity multiplied by the interval of time, or

$$x = t V \cos \phi. \quad (301)$$

In the vertical plane the projectile's travel is influenced by gravity as well as by the vertical component of its velocity. The vertical distance (y) covered by the projectile in any interval of time (t) is then the vertical component of the initial velocity multiplied by the interval of time, less the distance the projectile falls due to the force of gravity in the same interval of time, or

$$y = t V \sin \phi - \frac{1}{2} g t^2. \quad (302)$$

The above equations are called the *primary equations*, as from them all other deductions are made.

Elements at any point

303. Substituting for t in (302) its value as found from (301) we have

Equation of the trajectory in vacuum $y = x \tan \phi - \frac{gx^2}{2V^2 \cos^2 \phi}$ (303)

This is the equation of the trajectory in vacuum, and its form indicates that the trajectory is a parabola with vertical axis.

304. Since $\frac{dy}{dx} = \tan \theta$, we have, by differentiating (303),

Angle of inclination at any point $\tan \theta = \tan \phi - \frac{gx}{V^2 \cos^2 \phi}$ (304)

which gives an expression for the *angle of inclination* at any point in the trajectory.

305. By differentiating equations (301) and (302) with respect to time we obtain, respectively, the horizontal and vertical components of the remaining velocity at any time (t).

$$\frac{dx}{dt} = V \cos \phi; \quad \frac{dy}{dt} = V \sin \phi - gt.$$

By combining these components we get the remaining velocity at any point in the trajectory.

$$v^2 = \left(\frac{dy}{dt} \right)^2 + \left(\frac{dx}{dt} \right)^2 = (V^2 \sin^2 \phi - 2gt V \sin \phi + g^2 t^2) + V^2 \cos^2 \phi$$

and since $\sin^2 \phi + \cos^2 \phi = 1$, the above reduces to

$$v^2 = V^2 - 2gt V \sin \phi + g^2 t^2 = V^2 - 2g \left(t V \sin \phi - \frac{gt^2}{2} \right).$$

The expression in the bracket being the value of y from (302), we have finally

Remaining velocity at any point $v = \sqrt{V^2 - 2gy}$. (305)

Equation (305) shows that the remaining velocity is equal to the initial velocity when $y=0$, which occurs both at the origin and at the point of fall. Also, the remaining velocity has its smallest value where y has its greatest value, or at the summit; it follows that the velocity must decrease up to the summit and then increase until at the point of fall it has again reached its initial value.

306. The horizontal and vertical velocities at any point in the trajectory are, by definition, the horizontal and vertical components, respectively, of the remaining velocity at that point, whence

Horizontal and vertical velocities at any point $v_h = v \cos \theta; \quad v_v = v \sin \theta$. (306)

It is to be observed, however, that in a vacuum no force operates to diminish the initial value of the horizontal velocity, since the only force operating on the projectile, other than that initially imparted by the gun, is gravity, and the latter acts only in the vertical plane. Hence the horizontal velocity at any point in the tra-

jectory must be equal to the horizontal component of the initial velocity, or $v_x = V \cos \phi$.

307. The time of flight to any point in the trajectory is obtained directly from (301), and is

$$\begin{array}{l} \text{Time of} \\ \text{flight to} \\ \text{any point} \end{array} \quad t = \frac{x}{V \cos \phi}. \quad (307)$$

Elements at the point of fall

308. The value of x at the point where the trajectory again cuts the horizontal plane through the origin is the *horizontal range*. Since at this point $y=0$, we may obtain an expression for the horizontal range by substituting $y=0$ in (303). Factoring the resulting expression, we find

$$x \left(\tan \phi - \frac{gx}{2V^2 \cos^2 \phi} \right) = 0.$$

This gives for one value $x=0$, which is evidently the origin. For the other we may now use the appropriate symbol for the horizontal range.

$$\begin{aligned} \frac{gX}{2V^2 \cos^2 \phi} &= \tan \phi \\ X &= \frac{2V^2 \cos^2 \phi \tan \phi}{g} \end{aligned} \quad (308A)$$

which simplifies as follows

$$\begin{aligned} X &= \frac{2V^2 \cos^2 \phi \sin \phi}{g \cos \phi} = \frac{V^2 (2 \sin \phi \cos \phi)}{g} \\ \text{Horizontal} \quad X &= \frac{V^2 \sin 2\phi}{g} \end{aligned} \quad (308)$$

Equation (308) shows that the horizontal range increases with the angle of departure until the latter reaches 45° , while for further increases in the angle of departure the range decreases; the same range is given by either of two angles of departure, one as much greater than 45° as the other is less than 45° (for example, the same range is given by $\phi=60^\circ$ as by $\phi=30^\circ$). Also, for a given angle of departure the range varies as the square of the initial velocity.

Equation (308) may also be re-written

$$\sin 2\phi = \frac{gX}{V^2} \quad (308B)$$

in which form it provides an expression for the angle of departure required to obtain a given range. If the right-hand member is greater than unity it indicates that the given range is not obtainable with the given initial velocity.

309. By putting for x in (304) its value X for the point of fall, we may find the angle of inclination at that point, or the *angle of fall*. For convenience the expression for X from (308A) may be taken; using in place of θ the symbol ω for angle of fall, we now have

$$\tan \omega = \tan \phi - \frac{2V^2 \cos^2 \phi \tan \phi g}{V^2 \cos^2 \phi g} = \tan \phi - 2 \tan \phi$$

whence

Angle
of fall

$$\tan \omega = -\tan \phi. \quad (309)$$

This shows that the angle of fall is numerically equal to the angle of departure.

310. By substituting $y=0$ in (305) we obtain values of the remaining velocity for both origin and point of fall. The former is, of course, the initial velocity; the latter is the *striking velocity*, and (305) shows that the striking velocity equals the initial velocity.

Striking
velocity

$$v_s = V. \quad (310)$$

311. The *time of flight* to the point of fall is obtained by putting X for x in (307).

Time of
flight
to point
of fall

$$T = \frac{X}{V \cos \phi}. \quad (311)$$

Elements at the summit

312. By definition, the summit of the trajectory is the point at which the projectile ceases to ascend and commences to descend. At this point the projectile must then momentarily be in horizontal flight, and the angle of inclination (θ) of the curve at this point must be zero. Substituting $\theta=0$ in (304), and denoting the abscissa of this point by its special symbol x_s , we have

$$0 = \tan \phi - \frac{gx_s}{V^2 \cos^2 \phi}$$

whence

$$x_s = \frac{V^2 \cos^2 \phi \tan \phi}{g}.$$

Simplifying this expression in the manner already shown for (308A), and comparing the result with (308), we have

Horizontal
distance
to summit

$$x_s = \frac{V^2 \sin 2\phi}{2g} = \frac{X}{2}. \quad (312)$$

This indicates that the summit is at mid-range.

313. By substituting in (303) the value of x_s just found, we find the ordinate for the summit, or *maximum ordinate* (y_s).

$$y_s = \frac{V^2 \cos^2 \phi \tan^2 \phi}{g} - \frac{g V^4 \cos^4 \phi \tan^2 \phi}{2g^2 V^2 \cos^2 \phi}$$

which simplifies directly to

$$y_s = \frac{V^2 \sin^2 \phi}{2g}. \quad (313A)$$

The above may further be simplified by multiplying both numerator and denominator of the right-hand term by $\cos \phi$, after which this term becomes divisible by the expression for X from (308) and finally reduces to

Maximum
ordinate

$$y_s = \frac{X \tan \phi}{4}. \quad (313B)$$

A very convenient expression for the maximum ordinate in terms of time of flight is arrived at from the consideration that the height of the summit equals the height from which the projectile would fall in one-half the time of flight (T).^{*} This results directly in the expression

$$y_s = \frac{1}{2}g \left(\frac{T}{2} \right)^2 = \frac{gT^2}{8}. \quad (313C)$$

314. The time of flight to the summit is obtained by substituting the value of x_s for x in (307); it is, however, equally convenient to note that since $x_s = \frac{X}{2}$

Time of flight to summit	$t_s = \frac{T}{2}.$	(314)
--------------------------------	----------------------	-------

315. By putting for y in (305) the value of the maximum ordinate (y_s), an expression for the remaining velocity at the summit may be obtained. However, as has already been noted in article 312, the projectile is momentarily in exactly horizontal flight at the summit, hence the remaining velocity at the summit equals the horizontal velocity at that point. And since, as noted in article 306, the horizontal velocity at any point equals the horizontal component of the initial velocity, we have directly

$$v_s = V \cos \phi. \quad (315)$$

316. The deductions made above may now be summarized as follows: They all pertain to the trajectory in vacuum, but in some instances they remain approximately true also for the trajectory in air, as noted herein.

General characteris- tics of the trajectory in vacuum	1. The trajectory is a parabola with vertical axis; this is ap- proximately true also in air.
	2. At the origin the angle of inclination is identical with the angle of departure; at the summit it is zero; at the point of fall it is identical with the angle of fall and equal numerically to the angle of departure.

For the origin and summit these conclusions are equally true for the trajectory in air; the angle of fall in air is, however, materially greater than the angle of departure.

3. The remaining velocity decreases from its initial value until the summit is reached, and then increases until at the point of fall it becomes again equal to the initial velocity, i.e., the striking velocity equals the initial velocity. In air the remaining velocity also decreases in the ascending branch and increase in the descending branch, but the increase starts at a point beyond the summit; in air the striking velocity is always very materially less than the initial velocity.

4. The horizontal velocity remains constant throughout the trajectory, while the vertical velocity decreases in the ascending branch and becomes zero at the vertex and in the descending branch increases until at the point of fall it is equal (but opposite in direction) to its value at the origin. In air the horizontal velocity is constantly reduced; the vertical velocity vanishes at the summit and increases in the descending branch, as in vacuum, but does not again attain its initial value.

^{*} This deduction may be verified by equating the values of y_s given by (313A) and (313C) and solving for T . The resulting value of T may then be equated to the value of T from (311), and this equation in turn solved for X . The final result is an expression for X identical with (308).

5. The range increases with increase of angle of departure up to a maximum value of 45° for the latter, but for further increase in the angle of departure the range decreases. Also, the same range is given by two angles of departure, one of which is as much greater than 45° as the other is less than 45° . The first of these conclusions is approximately true also in air although, for reasons to be brought out later, with modern guns, projectiles, and velocities, the limiting angle is somewhat greater than 45° (about 50° for large guns). The second conclusion is true in air to the extent that for angles of departure greater than the limiting value (for air), the range is decreased, but the relation that has been stated for vacuum conditions holds only very approximately in air.

6. For a given angle of departure, the range varies as the square of the initial velocity. This is approximately true in air; actually, in air, range increases somewhat less than in proportion to the square of the initial velocity.

7. The summit, or vertex, of the trajectory is located at mid-range, and the time of flight to the summit is one-half the time of flight to the point of fall. These relations remain approximately true in air. For large guns the range to the summit, in air, varies from about .51 to about .53 of the total range, while for small guns the range to the summit may become almost .60 of the total range. The time of flight to the summit, in air, is generally between .46 and .49 of the total time of flight, and usually nearer the latter figure.

The theory of rigidity of the trajectory

317. Since in actual practice it usually occurs that the target is not in the horizontal plane through the gun, it is of interest to extend our study of the trajectory in vacuum to a comparison between a trajectory referred to an inclined plane and one referred to the horizontal plane. To do this we will seek a relation between the inclined range X' resulting from an angle of elevation ϕ' and angle of position p , and the horizontal range X resulting from an angle of departure ϕ equal to ϕ' .

In Figure 4 let x, y represent the coördinates of the point P on the trajectory OPH , and X' the inclined range to that point. The angle of elevation with respect to the line of position OP is $\phi' = \phi - p$, and the angle of position is defined

by $\tan p = \frac{y}{x}$. Equation (303) may be written

$$\frac{y}{x} = \tan \phi - \frac{gx}{2V^2 \cos^2 \phi}$$

whence

$$\tan p = \tan \phi - \frac{gx}{2V^2 \cos^2 \phi}$$

and

$$x = \frac{2V^2}{g} \cos^2 \phi (\tan \phi - \tan p).$$

The right-hand term of this expression may be expanded to

$$\tan \phi - \tan p = \frac{\sin \phi \cos p - \cos \phi \sin p}{\cos \phi \cos p} = \frac{\sin (\phi - p)}{\cos \phi \cos p}$$

whence,

$$x = \frac{2V^2 \sin(\phi - p) \cos \phi}{g \cos p} = \frac{2V^2 \sin \phi' \cos(\phi' + p)}{g \cos p}.$$

Since $X' = x \sec p$, we have finally

$$X' = \frac{2V^2 \sin \phi' \cos(\phi' + p)}{g \cos^2 p}. \quad (316)$$

Equation (316) is an expression for the inclined range X' resulting from an angle of elevation ϕ' with respect to a line of position making the angle p with the horizontal. Equation (308) is an expression for the horizontal range X resulting from an angle of departure ϕ . In order to compare the horizontal range with the inclined range for the condition $\phi = \phi'$, let us rewrite (308) with ϕ' in place of ϕ , and expand the term $\sin 2\phi'$.

$$X = \frac{2V^2 \sin \phi' \cos \phi'}{g}.$$

Dividing (316) by the above we have

$$\frac{X'}{X} = \frac{\cos(\phi' + p)}{\cos \phi' \cos^2 p}.$$

After expanding $\cos(\phi' + p) = \cos \phi' \cos p - \sin \phi' \sin p$, the above reduces to

$$\frac{X'}{X} = \sec p(1 - \tan \phi' \tan p). \quad (317)$$

318. The right-hand term of (317) becomes equal to unity when $p = 0$, which is to be expected since in that case X' and X are identical. If p is positive, the right-hand term has a value less than unity, which means that X' is less than X ; this is also to be expected since this situation represents the case of firing up an incline. Similarly, for the case of firing down an incline, X' becomes greater than X .*

But as long as p remains very small $\sec p$ does not vary much from unity and $\tan p$ does not vary much from zero; under these conditions the entire right-hand term of (317) does not vary much from unity. In naval gunnery we have ordinarily to deal only with very small angles of position, except in the case of antiaircraft fire. In surface fire it would be most unusual to encounter an angle of position greater than about $35'$, and a value as great as this occurs only at extremely short ranges, in which case ϕ' is also small. For example, for a $3''$ gun mounted 50 feet above the water line and firing at the water line of a target 1600 yards distant, the angle of position is about $(-)35'$ and the angle of elevation $48'$. The ratio

$\frac{X'}{X}$ in this case is 1.0001 and the variation quite negligible. For the same gun, at

5000 yards with p now reduced to $(-)11'$ and ϕ' increased to $5^\circ 17'$, the ratio becomes 1.0003; at 8500 yards, with $p = (-)7'$ and $\phi' = 15^\circ 35'$, the ratio becomes 1.0006. These examples represent rather extreme situations for fire from ships against surface targets. Under ordinary conditions the ratio will not vary appreciably from unity.

* These conclusions are also generally true in air as long as the angle of position does not become large. With large angles of position the reverse may become true. Ref. §38, *Handbook of Ballistics*, Vol. I, by Cress and Becker.

319. We may conclude from this investigation that with very small angles of position the range is not affected appreciably by the angle of position. This means that we may use the same angle of elevation whether the target is on the same level as the gun or on a different level, provided that we limit ourselves to such cases as are encountered in ordinary practice, *excluding antiaircraft fire*.

The assumption just made is known as the *assumption of the rigidity of the trajectory*. It means, in effect, that a given trajectory may be swung through a small vertical angle, as illustrated in Figure 5, without appreciably altering its form. It is important to note, however, that in any comparison of elements of trajectories under the terms of the above assumption, the elements must be referred

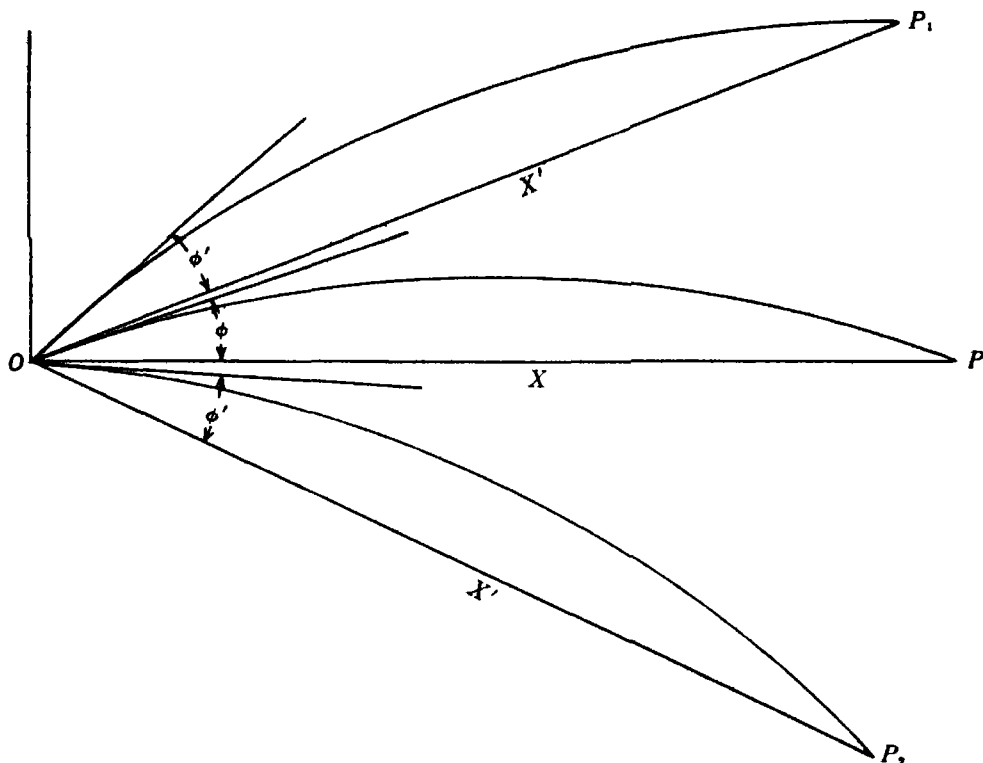


FIGURE 5

to their respective lines of position. For example, the maximum ordinates of the three curves in Figure 5 may be considered equal only if each is measured perpendicularly from the mid-range point of its own line of position; according to the usual definition of maximum ordinate, the values of this element are, of course, affected appreciably even by small angles of position. This applies to any elements which ordinarily are measured with respect to the horizontal plane. The immediate application of the assumption of rigidity of the trajectory, in actual service, is the use of the same sight-bar graduations for firing at targets which may be either slightly elevated or slightly depressed with respect to the gun, such as a gun on a high deck firing at a submarine, or a submarine's gun firing at the superstructure of a ship.

Although the foregoing assumption has here been based only on an investigation of conditions in vacuum, it is equally applicable in air, for the effects of air resistance on the quantities X and X' are so nearly equal as to cause no appreciable alteration in the ratio between these quantities.

CHAPTER 4

THE RETARDATION OF THE PROJECTILE DUE TO RESISTANCE OF THE ATMOSPHERE. THE COEFFICIENT OF FORM. THE BALLISTIC COEFFICIENT.

New Symbols Introduced

- R_f . . . Resistance of the air against the flight of a projectile (in lbs.).
 R_a . . . A general notation for the retardation of a projectile due to air resistance (in f.s.s.).
 A A constant used in Mayevski's retardation functions (art. 409).
 a An exponent used in Mayevski's retardation functions (art. 409).
 G_s . . . The retardation of a standard projectile, in air of standard surface density, according to the G -function (in f.s.s.) (art. 411).
 H_s . . . The standard relation assumed for the variation of air density with altitude, based on standard air density at the surface (art. 420).
 E The retardation of a given projectile in the standard atmosphere (in f.s.s.) (art. 429).
 w Weight of projectile (in lbs.).
 d Diameter of projectile (in inches).
 i Coefficient of form of a projectile (arts. 413-417).
 δ_s . . . The assumed standard for density of air at the earth's surface (1.2034 kg/m³).
 δ_a . . . The actual density of air at the earth's surface, under given conditions.
 δ The surface density factor, representing the ratio δ_a/δ_s .
 δ_b . . . The ballistic density factor, or ballistic density, for a given trajectory (art. 421).
 C The ballistic coefficient (art. 428).
 g The acceleration due to gravity (to be taken as 32.16 f.s.s. unless otherwise indicated).

Early theories 401. Although the effect of air resistance was not altogether dismissed by earlier investigators, Newton appears to have been the first to deal quantitatively with this factor. From purely theoretical considerations he determined that the retardation due to the air resistance must vary in proportion to the square of the velocity, provided that the effects of pressure diminution behind the projectile and of the air stream set up by friction might be neglected. He tested this principle by dropping spheres of various densities from the dome of St. Paul's Cathedral, and found it to hold approximately true for the comparatively low velocities thus attained. But subsequent experiments soon showed that the effects of air resistance on the flight of projectiles cannot be accounted for by anything so simple as Newton's quadratic law.

402. Robins was the first to measure the velocity of projectiles with tolerable accuracy. In 1742 he published his famous *New Principles of Gunnery*, where in he described his invention, the *ballistic pendulum*, and stated the results

The ballistic pendulum obtained therewith. The ballistic pendulum consisted of a heavy bob suspended from a tripod, arranged so as to receive the impact of a projectile and measure the resulting swing; from the velocity imparted to the bob, the striking velocity of the projectile was determined. With this device Robins measured velocities up to about 1700 f.s., and determined approximately the loss of velocity during flights up to about 250 feet from the gun. The results thus obtained were approximately in agreement with Newton's quadratic law for velocities up to about 900 f.s., but revealed great discrepancies between this law and observed results for higher velocities. Robins also suggested using the gun itself as a pendulum for determining the initial velocity, and Rumford soon thereafter experimented with such a device.

Hutton made extensive experiments in England from 1775 to 1791, using both the gun pendulum and the ballistic pendulum. In France, in 1839-40, the *Commission de Metz* conducted further extensive experiments with the ballistic pendulum, and from the results obtained established empirical formulas for the calculation of range and time of flight for various angles of departure.

403. Experiments with the ballistic pendulum ceased when, in 1840, Wheatstone suggested measuring the velocity of a projectile by causing it to cut successive wire screens, each of which formed part of an electric circuit containing an element for measuring time. **The chronograph** The *Commission de Metz* in 1856-57 undertook new experiments, using a chronograph designed by Navez, but the results did not have lasting significance. The experiments of an English clergyman, Bashforth, in 1866-70, using a chronograph designed by himself, were notable as being the first to lead to a fairly accurate analysis of the effects of air resistance. Bashforth not only measured air resistance with fair accuracy, but also showed how the resistance varied with different types of projectile head; among other things, he concluded that the resistance is proportional to the projectile's cross-sectional area. Many experimental programs followed soon after the development of the chronograph. The most important were: in France, the *Commission de Gâvre*, 1856-61, 1873, and 1888; in England, Bashforth, 1866-70, and 1878-80; in Russia, Mayevski, 1868-69; in Germany, Krupp, 1875-81; in Holland, Hojel, 1884. A most notable contribution in this period was the analysis made by Mayevski, and later extended by Zaboudski, of the results of the English, Russian, and German experiments.

Wheatstone's proposal that velocity might be measured by causing the projectile to interrupt electrical circuits, initiated the first real progress in the science of ballistics, and this principle has continued up to the present day to be an essential factor in the advancement of the science. The *Boulengé* chronograph became the most successful of instruments of its type and was used extensively until recently. The principle of interrupting circuits by causing the projectile to cut wire screens was later used with oscillographs. At the present time a variation of this principle is used, in the form of the so-called solenoid type of oscillograph, which depends upon electrical impulses generated as a magnetized projectile passes through wire loops placed in its path.*

404. As an illustration of a method by means of which the retardation of a projectile may be measured with the aid of a chronograph (or its equivalent), the following elementary description and example will suffice.

The gun is fired practically horizontally through screens located at such dis-

* For complete descriptions of the instruments here mentioned, see *Naval Ordnance*.

Experimental
determination
of retardation

tances from the muzzle that the portion of the trajectory intercepted by them is very nearly a horizontal, straight line. We may then consider, with negligible error, that the length of the trajectory between the points of measurement is equal to the horizontal distance between these points, and that the retardation measured between these points is not affected by the force of gravity and hence is due only to air resistance. Two pairs of screens and two chronographs are required; the distance between each pair, and the distance between the midpoints of the two pairs, must be known accurately. A chronograph is placed in circuit with each pair of screens, and as the projectile passes through them the time of flight between the screens of each pair is measured. The average velocity between each pair of screens is then determined by dividing the distance between the screens by the measured time of flight; for each pair this velocity may be considered to apply to the midpoint of that pair. We shall denote the velocity at the first pair by v_1 and at the second pair by v_2 , and the loss of velocity between the two pairs (i.e., between their midpoints) by $v_1 - v_2$. The average retardation for the flight between the two midpoints is found by dividing the loss of velocity by the time elapsed between these points; the latter may be determined (without appreciable error for the short distance involved) by dividing the distance l between the two midpoints by the average velocity $\frac{v_1 + v_2}{2}$ between these points. Then we have finally,

$$R_a = \frac{(v_1 - v_2)(v_1 + v_2)}{2l}. \quad (401)$$

The retardation thus determined applies to a velocity which is approximately the mean of the two measured, or $\frac{v_1 + v_2}{2}$ *.

In order to determine the total air resistance against the projectile at the same average velocity, we may apply the law of physics, $F = \frac{w}{g} a$. We shall let R_f denote the force of air resistance (in lbs.). R_a as already determined is the acceleration (or deceleration in this case); w is the weight of the projectile. Then

$$R_f = \frac{w}{g} R_a. \quad (402)$$

An application of the above formulas is given in the following example.

Given: A 6" projectile weighing 105 lbs. was fired practically horizontally through two pairs of screens. The distance from the gun to the midpoint of the first pair was 400 feet, and to the midpoint of the second pair 915 feet. The velocity, as measured by chronograph, was 2221 f.s. between the first pair and 2173 f.s. between the second pair.

* By firing through several pairs of screens the rate of change of retardation may be determined and the correspondence between retardation and velocity more accurately defined. A complete treatment of this problem is given on pp. 51-59, *New Methods in Exterior Ballistics*, by F. R. Moulton.

Find: What was the average retardation between the points of measurement, and to what velocity does it apply (approximately)? What was the total resistance encountered by the projectile at this velocity?

For substitution in (401), $l = 915 - 400 = 515$ feet, $v_1 + v_2 = 4394$ f.s., and $v_1 - v_2 = 48$ f.s.

$$\begin{array}{rcl}
 (v_1 - v_2) = 48 & \dots\dots\dots & \log 1.68124 \\
 (v_1 + v_2) = 4394 & \dots\dots\dots & \log 3.64286 \\
 2l = 1030 & \dots\dots\dots & \log 3.01284 \quad \text{colog } 6.98716 - 10 \\
 R_a = 204.76 \text{ f.s.s.} & \dots\dots\dots & \log 2.31126
 \end{array}$$

With formula (402) we find,

$$\begin{array}{rcl}
 R_a = 204.76 & \dots\dots\dots & \log 2.31126 \\
 w = 105 & \dots\dots\dots & \log 2.02119 \\
 g = 32.16 & \dots\dots\dots & \log 1.50732 \quad \text{colog } 8.49268 - 10 \\
 R_f = 668.54 \text{ lbs.} & \dots\dots\dots & \log 2.82513
 \end{array}$$

The retardation of 204.76 f.s.s. and resistance of 668.54 lbs. apply to the mean velocity 2197 f.s. (approximately).

405. The nature of the results obtained in the various experiments to measure the retardation due to air resistance soon indicated that the factors entering into this retardation are exceedingly complex. Practically all of the earlier attempts to reconcile the observed results with theory clung to the quadratic law as a general basis and sought to bring about agreement, under the terms of a quadratic law, by supplying the latter with coefficients varying for different velocity bands. Even this expedient failed to yield satisfactory agreement and it eventually was accepted that retardation varies according to a variable power of the velocity. An examination of the phenomena attending the flight of a projectile will serve to explain some general features of the laws that eventually were developed.

406. Photography of projectiles in flight has been an invaluable aid in revealing some of the most important of these phenomena. With suitable lighting arrangements photography can be made to show very clearly the nature of the disturbances of the air medium surrounding a projectile in flight. Plate I* shows three photographs of an 8 mm. bullet (approximately *. 30 cal.) in flight through air at about 2900 f.s. The conditions for all three of these photographs were the same, the differences in the appearance of the air disturbance being caused by differences in lighting. The remarkably clear revealment of the waves issuing from the head and tail, and of the turbulence to the rear of the projectile, is brought about by refraction of light through the condensations and rarefactions of the air at these points.

* Plates I, II, and III, have been reproduced from *Handbook of Ballistics*, Volume I, Cranz and Becker, by courtesy of the publisher (Julius Springer, Berlin). For additional photographic records, and description of details of making the records, see the above reference; also, *Spark Photography and Its Application to Problems in Ballistics*, by Philip P. Quayle (Scientific Papers of the Bureau of Standards, No. 508); also, *A Camera for Studying Projectiles in Flight*, by H. L. Curtis, W. H. Wadleigh, and A. H. Sellman (Technological Papers of the Bureau of Standards, No. 255); also *Lehrbuch der Ballistik*, Volume III, Cranz (5th Edition).

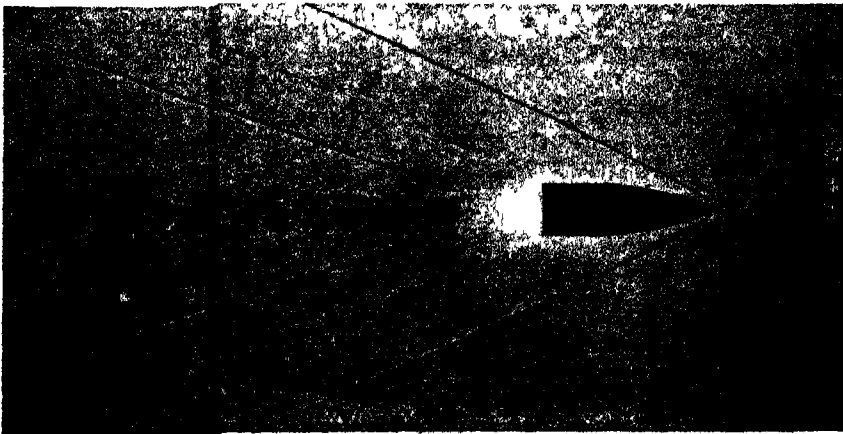
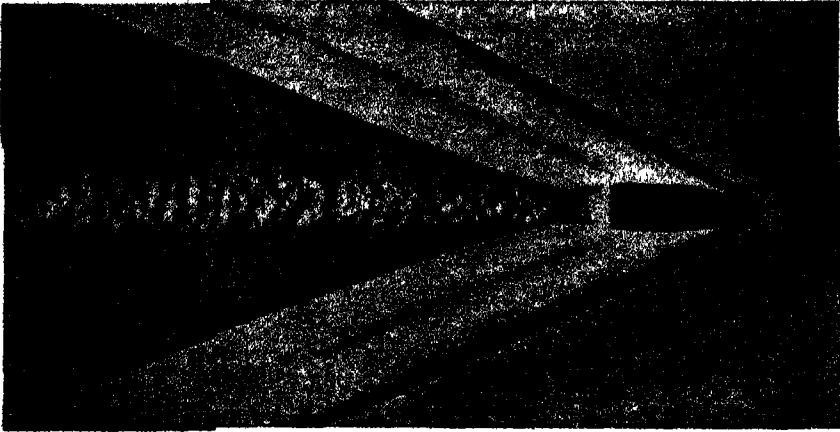


PLATE I

The sharply pointed wave front shown in these pictures may be accounted for as follows. In Figure 6 let the line AE represent the line of flight of a projectile, and E be the position of the tip of the projectile at the instant depicted. At E the tip of the projectile is just impacting on the air at that point, and the resulting disturbance will travel as a spherical wave of condensation centered at E (as a stone thrown into water sets up a circular ripple). The same thing has already happened at earlier points in the flight of the projectile, as at A , B , C , and D , and we therefore find each of these points now the center of a wave spreading

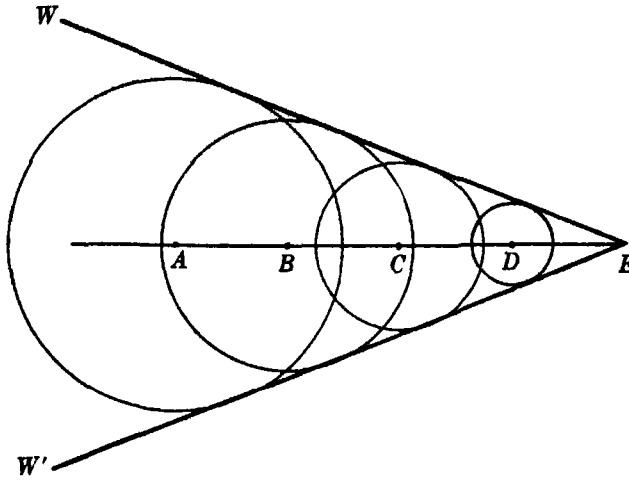


FIGURE 6

outward. The waves are receding from their respective initial points of disturbance at the velocity of sound, and the radii at the several points are proportional to the intervals of time elapsed since the projectile passed them. Under the assumed construction, a line drawn from E tangent to any one of these spheres is tangent to all of them, as shown by EW and EW' . Considering an infinitely great number of successive positions of the projectile between A and E , the peripheries of the resulting waves evidently form solid lines EW and EW' , and these lines then form the wave front shown in the photographs. In the upper photograph of Plate I the wave front of the tail wave is seen to be an envelope for such circular waves. A particularly vivid demonstration of this feature is shown in Plate II. This is a photograph of the air disturbance attending the flight of a bullet (approximately ".30 cal., at 2900 f.s., as in Plate I) through a perforated

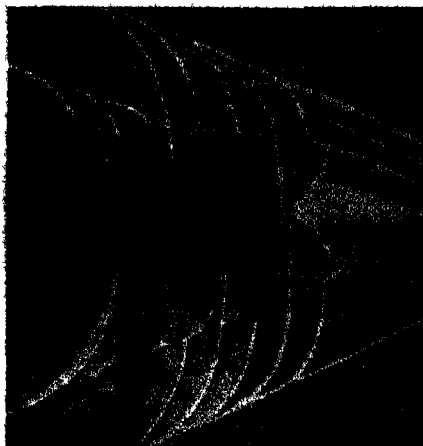


PLATE II

waves' outward motion, and hence the more acute will be the angle of the wave front. The slope of the wave front is, indeed, a measure of the projectile's velocity compared to the velocity of sound, but irregularities of the latter under the abnormal conditions in the immediate vicinity of the projectile prevent it from being used as an accurate measure in this connection.

Let us consider now what should happen if the projectile's velocity is less than the velocity of sound. In Figure 7 let AE again represent the line of flight of a projectile, but in this case a projectile traveling at a velocity less than that of sound.

By the time the projectile has reached point E , the wave initiated by it at point A will have preceded it to A_1A_2 , and similarly the waves initiated at B , C , and D will have preceded it as shown. Plate III also illustrates this case; it is a photograph of a bullet travelling at about 1115 f.s., which, under the attending conditions, was somewhat less than the velocity of sound. Here the wave front has disappeared and we see, instead, a detached wave of condensation in advance of the projectile.

408. From theoretical considerations, as well as from the indications obtained photographically, it appears that the loss of energy of a projectile in flight is due to the following effects. No attempt is made here, however, to evaluate the proportionate amounts of these effects.

tube. The purpose of this arrangement was to separate the waves, by permitting only a portion to escape through the holes in the tube. By suitable lighting effects these escaping waves were then shown, as they emerged from the tube, combining to form the wave front as before.

407. The shape of the wave fronts shown in Plates I and II and in Figure 6 depends, of course, on the fact that the velocity of the projectile is greater than the velocity of sound (i.e., greater than the velocity at which the waves of disturbance travel in air). The greater the excess of the projectile's velocity over the velocity of sound, the greater will be the ratio between the projectile's forward motion and the

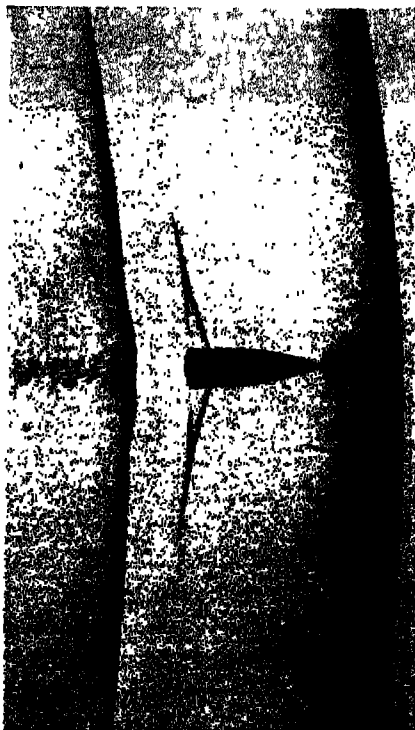


PLATE III

1. Energy is dissipated in the creation of air waves. The amount of this effect evidently is influenced by the form of the projectile, as well as by the area of its cross section.
2. Energy is dissipated in the creation of suction and eddy currents; the marked turbulence in the wake of the projectile (Plates I, II, and III) gives some evidence as to the degree of this effect. This effect evidently is influenced chiefly by the form of the projectile, and particularly by the form of its after-body.
3. Energy is dissipated in the form of heat generated by frictional resistance. This effect evidently is influenced by the form of the projectile, as well as by the area and character of its surface.

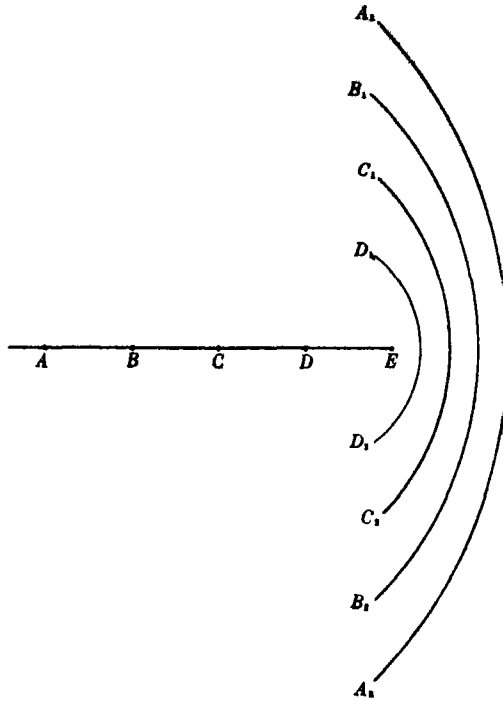


FIGURE 7

Although sufficient knowledge to permit separate evaluation of these several effects is not at hand, the indications are that the first two enumerated probably account for most of the total loss of energy suffered by the projectile. The loss of energy due to the creation of waves is minimized by giving the projectile a smooth contour, for points of sharp change in contour are sources of additional wave disturbances. Plates I and III show that the rotating band and the rear face of the projectile, as well as its tip, are sources of wave disturbances. The suction and eddy effects are influenced materially by the shape of the projectile's after-body. Considering only the problem of minimizing air resistance, the shape of the projectile should be very much like that of an airship. However, other considerations prevent anything more than an approximation of such a shape; the need

for rotating bands and for uniform passage through the bore calls for a cylindrical body and prevents use of the streamline shape.*

409. Consideration of the wave phenomena revealed by photographic records, and of their explanations as outlined above, must lead at once to the conclusion that the relation between velocity of projectile and velocity of sound has a most important bearing on the resistance offered by the air to a projectile's flight. Furthermore, considering the very marked change in the condition of affairs according to whether the projectile's velocity is above or below the velocity of sound, it is very reasonable to suppose that any law which might be set up to account for observed results must be capable of showing a very marked, perhaps even abrupt, change at the velocity of sound. Indeed, one might even suspect that perhaps one law should be sought for velocities less than the velocity of sound and a totally different one for velocities greater than the velocity of sound.

In the search for an adequate and yet reasonably simple expression of a retardation law, quadratic, cubic, and bi-quadratic laws, and combinations of these, were tried, but none of these remained long in favor. Mayevski appears to have been the first to introduce the idea of establishing zones of velocities, with different exponents for each zone. On the basis of the English and Russian experiments of 1866-70 he established three zones, viz., 0-280 m.s., 280-360 m.s., and 360-510 m.s.; for the first of these zones he assumed resistance to vary according to a combination of square and fourth power of velocity, for the second according to the sixth power, and for the third according to the square. Hojel, from his own experiments of 1884, established five zones, viz., 140-300 m.s., 300-350 m.s., 350-400 m.s., 400-500 m.s., and 500-700 m.s., and for these zones he assumed, respectively, the exponents 2.5, 5, 3.83, 1.77, and 1.91.

The most widely accepted series of laws, or functions, was that given by Mayevski in 1881, and extended later by Zaboudski; this series of functions was based on the Krupp firings of 1875-81, as well as on Mayevski's own experiments of 1868-69 and Bashforth's of 1866-70. Mayevski's functions have until quite recently been accepted as standard for ballistic computations in our own services, and they are even at present used by us to a limited extent. They are represented by the general expression

$$R_a = \frac{A}{C} v^a \quad (403)$$

in which R_a denotes the retardation corresponding to the velocity v , and the coefficient A and exponent a vary from zone to zone but remain constant within a given zone. The ballistic coefficient, C , combines factors pertaining to the weight, cross-section, and form of the projectile, and to the atmospheric density, all of which will be discussed presently. However, it is appropriate to note here that the standard of form assumed by Mayevski was that of a projectile about three calibers long with an ogival head rounded to a two-caliber radius. Mayevski's functions, as translated into English units by Col. Ingalls (U. S. Army), are given below:

* A slight tapering at the rear end of the projectile, known as "boat-tailing," has been applied successfully to projectiles up to about 6" caliber.

v between 3600 f.s. and 2600 f.s.

$$R_a = \frac{A_1}{C} v^{1.55} \quad \log A_1 = 7.60905 - 10$$

v between 2600 f.s. and 1800 f.s.

$$R_a = \frac{A_2}{C} v^{1.7} \quad \log A_2 = 7.09620 - 10$$

v between 1800 f.s. and 1370 f.s.

$$R_a = \frac{A_3}{C} v^2 \quad \log A_3 = 6.11926 - 10$$

v between 1370 f.s. and 1230 f.s.

Mayevski's
retardation
functions

$$R_a = \frac{A_4}{C} v^3 \quad \log A_4 = 2.98090 - 10$$

v between 1230 f.s. and 970 f.s.

$$R_a = \frac{A_5}{C} v^5 \quad \log A_5 = 6.80187 - 20$$

v between 970 f.s. and 790 f.s.

$$R_a = \frac{A_6}{C} v^3 \quad \log A_6 = 2.77344 - 10$$

v between 790 f.s. and 0 f.s.

$$R_a = \frac{A_7}{C} v^2 \quad \log A_7 = 5.66989 - 10$$

410. In recent years it has become the practice to abandon all formal expressions of a velocity-retardation relation, or series of such expressions, and to resort to explicit use of a tabulated relation. The latter practice minimizes compromise between experiment and theory, insofar as the velocity-retardation relation is concerned, for it makes practically direct use of observed data. Reduction of observed results to a convenient system of standard conditions represents, of course, the application of theory to some degree. Any tabulated function may be expressed also by a single analytical equation, by making the latter as complicated as necessary. In the case of the velocity-retardation function, such an expression becomes exceedingly complicated and totally worthless for any purposes of direct application to solutions of the trajectory. Such expressions have been developed, but their application is confined to extrapolation beyond the limits of observed values, in tabulating the function.*

The results obtained by the *Commission de Gâvre* in its experimental program of 1888 were reduced to a tabular velocity-retardation function that came to be known as the *Gâvre function* and to be accepted extensively in the United

* A single analytical expression closely approximating Mayevski's functions was set up by Siacci, but it contained some 17 factors in complicated arrangement; this expression is given on page IV of *Artillery Circular M* (1917), U. S. Army. The single expression representing the *Gâvre function* is of similar complexity; it may be found on page VI of *Exterior Ballistic Tables Based on Numerical Integration*, Volume I (U. S. Army, 1924). The latter reference states also the sources of the original *Gâvre function* and of its modifications to the form now in use.

The G-function

States as well as abroad. In the United States a slightly modified version of the Gåvre function, referred to generally as the *G-function*, is used in connection with present-day computations of trajectories, and the numerical illustrations and exercises presented in this text will be confined to the use of the latter.

411. The *G-function* may be thought of as a tabular expression of the relation between retardation and velocity, that is, as a table from which may be found directly the retardation corresponding to any given velocity. Denoting retardation (according to the *G-function*) by G_v , and velocity by v , this amounts then simply to a tabulation of G_v against the argument v . We will refer to this tabulation as the *G-table** (Table I, *Range and Ballistic Tables, 1935*).

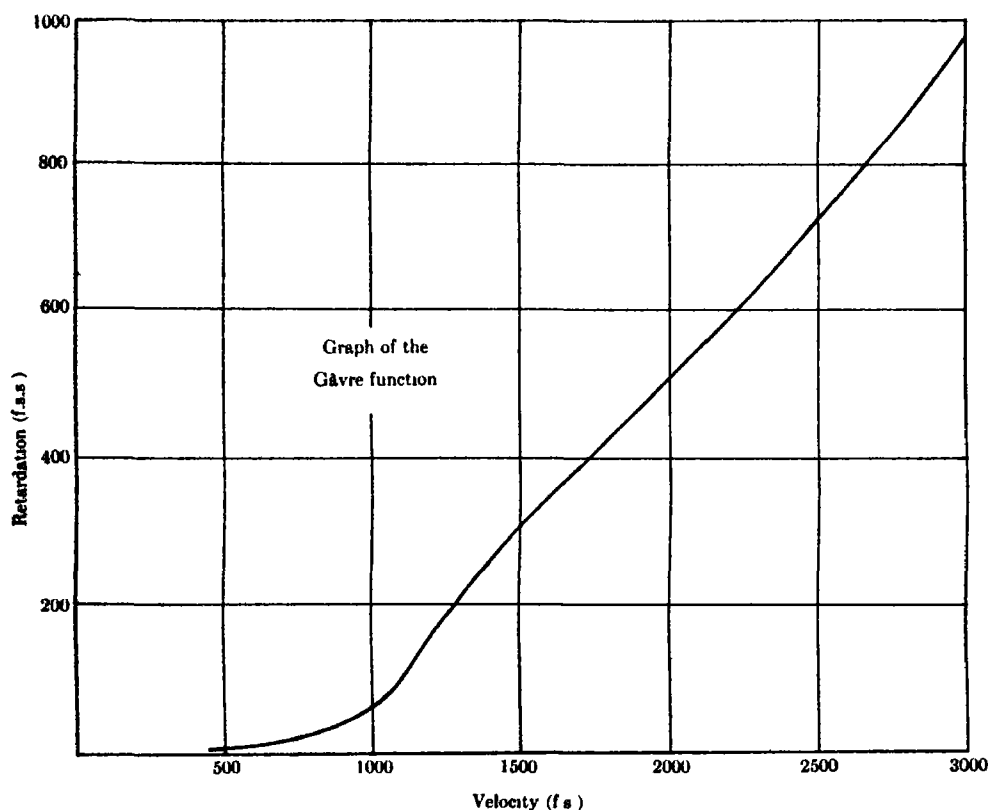


FIGURE 8

The graph of the *G-function* is shown in Figure 8. It is of particular interest to note the marked change in the character of the curve as it approaches the velocity of sound (about 1100 f.s.), and also the peculiar inflexions between about

* Since in many methods actually in use for computing trajectories the expressions $\frac{G_v}{v}$ or $\frac{G_v}{v^2}$ occur frequently, tables of the *G-function* usually are in terms of these expressions rather than in terms of G_v directly (as here defined). Also the entering argument is usually $\frac{v^2}{100}$, for a similar reason. In all discussions and exercises in this text, however, the term G_v is meant to denote retardation directly, and in the *G-table* employed it is tabulated against v directly.

1000 f.s. and about 1500 f.s. (compare this with the behavior of the exponent of Mayevski's functions in the same region). This curve then bears out what already has been suspected from an investigation of the photographic records of Plates I, II, and III. The graph of Mayevski's functions, although somewhat less continuous, agrees closely with the *G*-curve; the greatest discrepancy occurs around 2500 f.s., where it is slightly greater than 3% (in terms of the retardation argument). Recent experiments (1926-33) conducted by the U. S. Army, using several types of projectile varying materially among each other as well as from the type used in the Gâvre experiments, show even more remarkable results.

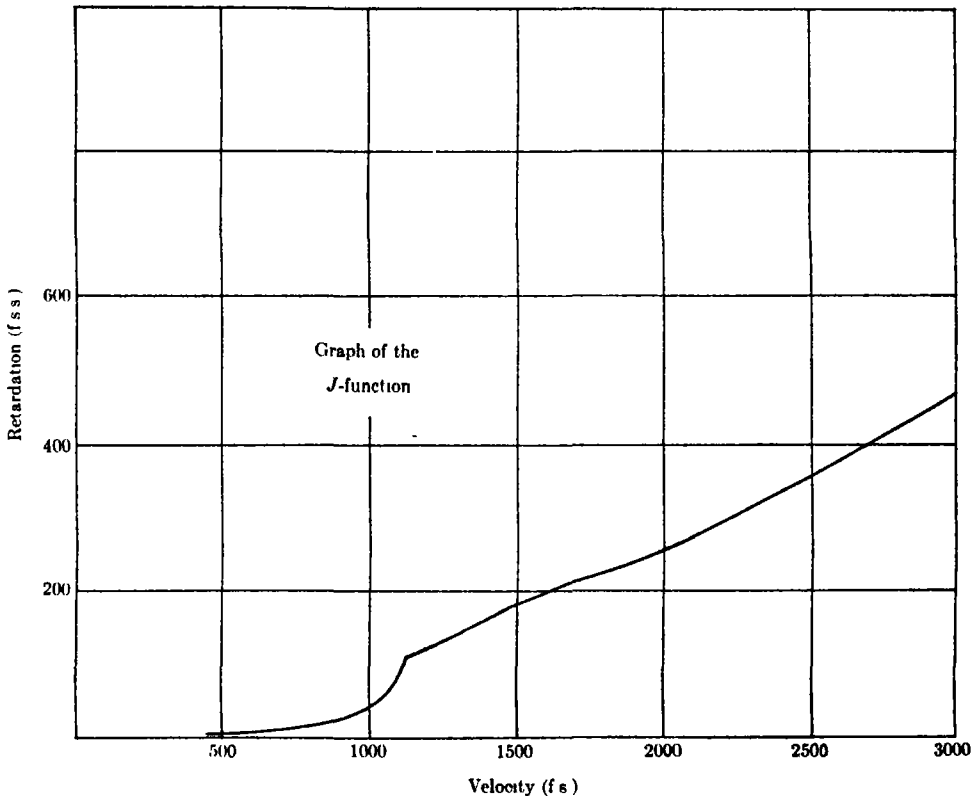


FIGURE 9

The graph of the velocity-retardation function obtained with one of these types is shown in Figure 9 (labeled *J*-function, this being the term used by the U. S. Army for the recently determined functions). It is to be noted that the latter shows an abrupt change practically at the velocity of sound, and that the branches of the curve lying on either side of this velocity are very dissimilar. These features are characteristic of all of the recently determined functions.

412. In addition to being a function of velocity, retardation evidently must depend also on certain properties of the projectile and of the resisting medium.

We shall now examine these in turn.

**Effect of
weight
and diameter
of projectile**

It is a well-known physical law that, other things remaining equal, the retardation suffered by a moving body in a resisting medium is inversely proportional to the body's weight; we may state, then, that *the projectile's retardation varies inversely as its weight w .*

It is generally assumed that, other things remaining equal, retardation is directly proportional to the cross-sectional area of the moving body (taken normal to the direction of its motion). Since the area of the projectile's cross-section is directly proportional to the square of the projectile's diameter d , we may state that the projectile's retardation varies directly as the square of its diameter, or d^2 .*

413. We know also that the retardation of a projectile depends upon its form, and to account for this we adopt a factor which is called the *coefficient of form* and denoted by the symbol i . For reasons already outlined above (art. 408) we cannot expect to assign to the coefficient of form a value based entirely on a direct measure of dimensions of the projectile. Certain measurable features of a projectile are found to constitute a basis of comparison among projectiles of closely similar shape, but for reasons that will appear presently the coefficient of form is treated wholly empirically and its measure is obtained wholly from actual performance.†

The coefficient of form features of a projectile are found to constitute a basis of comparison among projectiles of closely similar shape, but for reasons that will appear presently the coefficient of form is treated wholly empirically and its measure is obtained wholly from actual performance.†

The coefficient of form is so chosen as to express directly the proportion between the retardation of a given projectile and that of an arbitrarily chosen *standard projectile* (other things remaining equal), so that the projectile's retardation varies directly as its coefficient of form i . It is to be noted that this coefficient is entirely relative, and that a projectile may have any number of coefficients of form, a different one for each different standard projectile to which it may be compared. When the G -function is used, as will be the case in this text, the coefficient of form represents a comparison with respect to the standard projectile assumed for that function, viz., a projectile about three calibers long and with an ogival head rounded to a two-caliber radius (the Gâvre standard did not differ appreciably from that assumed by Mayevski).

414. Considering the manner in which the coefficient of form is to be determined, it follows that the value obtained for this coefficient must represent not only influences depending purely on the shape of the projectile, but also any other influences that have not been accounted for separately. For example, let us assume that by an experimental firing we have actually observed that at 2500 f.s. a certain projectile having a weight of one pound and a diameter of one inch suffers a retardation of 362 f.s.s. Also that the G -table (art. 411) states that at

2500 f.s. the retardation of the standard projectile is 724 f.s.s. Since $\frac{w}{d^2}$ equals

unity for the projectile fired, it is necessary to assign to it the value $i = .500$ in order to account for the fact that its observed retardation is only one-half that of the standard. It should be clear, however, that any false assumptions in the

* Some authorities disagree with this assumption and cite experimental results which indicate that for projectiles of large cross section the resistance per unit of area is less than the resistance per unit of area for projectiles of small cross section. (Ref. pp. 36-38, *Handbook of Ballistics*, Crans and Becker). The assumption given above is, however, commonly adopted both here and abroad. Any discrepancies that may result from it are, of course, reflected in the coefficient of form.

† According to the French ballisticians, Hélie, the coefficient of form of an ogival pointed projectile varies in direct proportion to the sine of the semi-ogival angle of the head, i.e., the angle between the projectile's axis and the tangent to the ogive where it intersects this axis. In French methods this relation is used, and the coefficient of form is a function of the semi-ogival angle, viz., $\sin \gamma$. In this country, however, the coefficient of form merely expresses a ratio of comparison between a given projectile and a standard projectile, and this ratio is not derived from any single feature of the projectile but from the actual performance of the projectile as a whole.

entire scheme of arriving at the conception of a coefficient of form, as well as all inaccuracies involved in the determination, will be reflected in the value obtained. The significance of this will become more evident as we proceed to the practical methods of computing trajectories.

415. The artificiality of the coefficient of form is readily seen if we seek to evaluate it, for the same projectile, according to more than one velocity-retardation function. Continuing with the above example we find, according to Mayevski's functions, that at 2500 f.s. the retardation of the standard projectile is 746 f.s.s. Since the observed retardation of the given projectile is actually 362 f.s.s.

at 2500 f.s., we must, in order to secure agreement with Mayevski's functions, assign to this projectile a coefficient of form equal to $\frac{362}{746}$,

or $i = .485$. The difference between $i = .500$ as obtained according to the G -function, and $i = .485$ as obtained according to Mayevski's functions, evident-

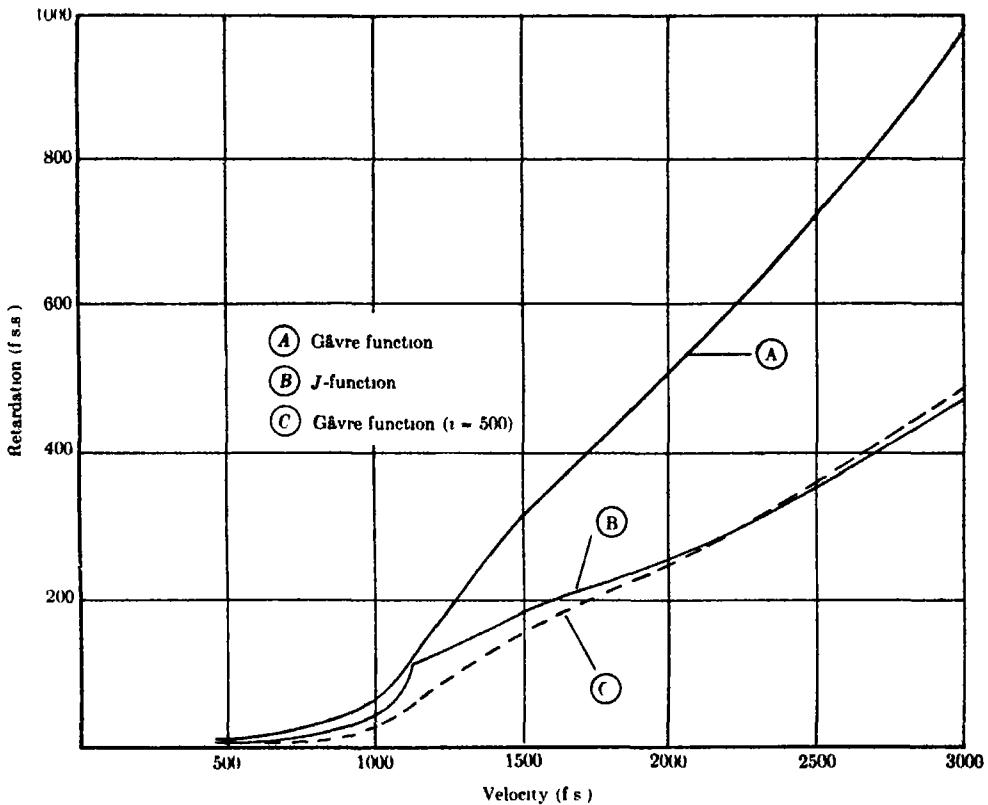


FIGURE 10

ly cannot be due to the projectile itself (since the same projectile at the same velocity has been considered in both cases); it is due entirely to disagreement between the Gåvre and Mayevski functions at 2500 f.s.

In order further to illustrate this important feature of the coefficient of form, let us examine the difference, in terms of coefficient of form, between the G -function and J -function; for this purpose we will refer to the graphs of these functions, which already have been shown in Figures 8 and 9, respectively, and in Figure 10 are shown drawn to the same scale. Curve A in Figure 10 is the graph of the G -function unmodified, i.e., it represents the performance of the

Gâvre standard projectile as defined by the G -function. Curve B similarly represents the performance, as defined by the more recent J -function, of a different standard projectile which we will call the J -projectile.

The very great discrepancy between the two curves is due first of all to the fact that they represent different projectiles. The J -projectile is about 5 calibers long, with a conical head about 2 calibers long and tapering almost to a point, and with the rear end "boat tailed" for a length of about one-half caliber from the rear face of the projectile; the resistance suffered by it therefore is materially less than that suffered by the comparatively blunt, square-tailed Gâvre projectile. Let us see now whether by means of a single coefficient of form we can make the Gâvre function account for the actual performance of the J -projectile. Assuming that the J -projectile has a coefficient of form $i = .50$ (as compared to the Gâvre projectile), and on this basis determining its retardation from the Gâvre function, we can plot a graph showing what the retardation of the J -projectile should be according to the G -function, and compare it with the graph of the J -function which represents the actually measured retardations of the projectile. This new graph is labeled C in Figure 10.

The value $i = .50$ evidently is exactly correct for a velocity somewhere between 2200 f.s. and 2300 f.s. For greater velocities than this we find that $i = .50$ is somewhat too great, and at 3000 f.s. the value $i = .48$ is required to bring about the desired agreement. From about 2200 f.s. to about 1120 f.s. the necessary value of i increases, becoming .60 at about 1500 f.s., and almost unity at about 1120 f.s. Below about 1120 f.s. the value decreases to about $i = .75$ at about 900 f.s., and for still lower velocities it increases again, becoming equal to unity at about 500 f.s. and remaining at about unity for very low velocities.

416. The above investigation amounts practically to a comparison between the actually observed retardations of two different projectiles (each reduced to the common basis of $\frac{w}{d^2} = 1$), and it shows that the relation between the two

cannot be represented by a constant, and hence that the coefficient of form which we have chosen to represent this relation is not a constant. It is important that this be borne in mind in connection with the methods of solution of trajectories that will be developed presently. The above investigation represents greater extremes than are met ordinarily in our own immediate problem, since the projectiles used by our Navy do not differ as much from the Gâvre projectile as does the J -projectile on which the above comparison is based.

417. A significant feature of the above investigation is the fact that the coefficient of form, as a factor of comparison between projectiles, may vary materially according to the velocity. In the illustration offered above the variations of i with velocity are heightened because of the relatively great dissimilarity between the projectiles considered, but variations of like nature, although of less amount, are found even when more nearly similar projectiles are considered. The reason for this is to be found in the existence of numerous influences attending the flight of a projectile, which are not fully expressible in terms of the factors chosen to make up the ballistic coefficient, nor, indeed, in terms of any factors which might be added thereto without greatly complicating matters. Let us consider, for example, the position (or attitude) of a projectile in flight. It is known that the axis of a projectile fired from a rifled gun oscillates about the tangent to the trajectory.* For a well-designed pro-

Causes of
variations
in i

* The motion of the projectile with respect to the tangent will be studied further in Chapter 9.

jectile these oscillations are of small amplitude and the axis remains always close to the tangent, but nevertheless the projectile is always somewhat oblique to the direction of its flight, rather than exactly head-on. This situation invalidates to some degree our assumption that the cross-sectional area opposed to the resisting medium, and the retardation resulting therefrom, are proportional to σ^2 . The effective area evidently depends also on the obliquity of the projectile with respect to its direction of flight.

The obliquity of a projectile to its direction of flight at any point in its trajectory depends, in general, upon its stability at that point and hence is influenced by all of the factors affecting stability. These, in addition to constant physical features of the projectile, include its rate of spin and its velocity. For a given stability the obliquity depends also on the rate of change of curvature of the trajectory at the point in question. Since no other factor separately accounts for any of these influences, the coefficient of form evidently must include all of them. The dependence of the coefficient of form on velocity is readily apparent from the above. Also, for the same projectile and same initial velocity we may expect the coefficient of form to vary with angle of departure, since both the curvature and the limits of velocity included within a given trajectory are influenced by the angle of departure.

The effect of obliquity to the direction of flight is not confined to an alteration of the effective cross-sectional area of the projectile. Unequal pressures on the sides of the projectile cause variations in flight. This is well evidenced by the drift, which is the result of lateral components due to obliquity; obliquity may equally well give rise to vertical components, and affect the range in this manner as well as through its influence on the retardation. It is to be appreciated, therefore, that the coefficient of form is influenced by many characteristics of the entire trajectory, and that *its evaluation ultimately must depend upon measurements applied to entire trajectories.*

418. Next to be considered are variations in the resisting medium itself, i.e., in the atmosphere; these may be classified as follows:

- | | |
|------------------------------------|--|
| Variations
in the
atmosphere | <ul style="list-style-type: none"> (a) Variations in the density of the atmosphere near the surface of the earth, as determined by the air temperature and barometric pressure near the surface. (b) Variations in the density of the atmosphere with altitude, due to the variations in temperature and pressure which are incident to changes in altitude. (c) Variations in the density of the atmosphere due to variations in its saturation with moisture (i.e., in its relative humidity), which depend both on surface conditions and on altitude. (d) Variations in the elasticity of the atmosphere, due to variations in its temperature alone, which depend both on surface conditions and on altitude. |
|------------------------------------|--|

It will be noted that the variations enumerated under (a), (b), and (c) above are of the same kind, i.e., they are all variations in the *density* of the atmosphere, although they result from different causes. A difference in the cause of a given density variation occasions no difference in its effect upon retardation; that is to say, a given percentage variation in air density has a like effect on retardation whether the variation be due to one or another of the above causes. The above distinction is made, however, in view of differences in procedure of taking into account the density variations due to the different causes enumerated. On the other hand, it is to be noted that the elasticity effect mentioned under (d), which depends upon the air temperature just as do the density effects (a), (b), and (c),

is nevertheless entirely independent of the latter; i.e., the air temperature affects density and elasticity both simultaneously but each independently. The effects on the retardation due to these several variations in the atmosphere, and the methods by which they are taken into account, will now be examined in turn.

419. It is generally supposed that the retardation of a projectile varies directly in proportion to the density of the air; this theory never has been substantiated by experimental proof, but it is, nevertheless, commonly accepted by ballisticians. Under this assumption it is possible to express the ratio between retardations under different conditions of air density, by means of a factor which expresses the ratio between the densities themselves. In other words, a given percentage variation in air density is assumed to cause a like percentage variation in retardation.

For tabulated retardation functions, a standard surface air density is assumed. Practice varies somewhat as to the choice of this standard density, but for tables used in the United States the standard is defined by the density of air at a temperature of 15°C. (59°F.), at a barometric pressure of 750 mm. (approximately 29.53 in.) of mercury, and 78 per cent saturated with moisture. The actual weight of air under these conditions is 1.2034 kilograms per cubic meter (approximately .075 pounds per cubic foot).^{*} We will denote by the symbol δ_s this standard surface density, by δ_a the actual surface density under any given set of conditions, and by δ the ratio of actual surface density under any given set of conditions to the standard surface density, i.e.,

$$\delta = \frac{\delta_a}{\delta_s} \quad (404)$$

This surface density factor, δ , therefore states the percentage of the standard surface density that is represented by the actual surface density under any given set of conditions. It follows that if $\delta > 1$, the actual density is greater than the assumed standard, and the retardation under the given conditions is greater than under standard conditions, i.e., greater than the tabulated value. We may state, then, that *retardation varies directly as δ* .[†]

Values of the ratio δ are tabulated in Table III,[†] *Range and Ballistic Tables, 1935*, against the arguments temperature (F.°) and barometer (inches of mercury), and for a constant saturation of 78%. Examination of this table will show that the value $\delta = 1$ corresponds to a great variety of combinations of temperature and pressure, as well as to the combination 59° and 29.53 inches. This is due to the fact that the density of air varies independently according to both temperature and pressure; hence the effect on density of a change in temperature may be balanced exactly by an accompanying change in pressure. On the other hand, variations of considerable magnitude may result from combinations of low temperature and high pressure, or vice versa, and may have a marked effect on the trajectory.

^{*} A discussion of the various standards that are, and have been, assumed both here and abroad, is given on pp. 15-18, *T.S. No. 148*, April, 1921, (U. S. Army).

[†] In the 1926 and 1930 editions of this book, the density factor δ was chosen to represent the ratio δ_s/δ_a , i.e., the reciprocal of the above, in conformity with tables which were used in connection with those editions. The present practice, both in the U. S. Army and in the U. S. Navy, is to use tables which give the ratio of actual density to standard density, or the value of δ as defined above. The only difference occasioned by this change of procedure is that δ now appears in the denominator of the ballistic coefficient (406), instead of in the numerator as heretofore.

429. The density of the atmosphere at a level above the earth's surface depends both upon the density at the surface and upon the altitude of the given level, since both temperature and pressure decrease with altitude. However, it is not a simple matter to define the density for a level aloft in terms of surface density and altitude, since it is known that fluctuations in the surface air density often are caused by local influences which do not extend very far aloft. Similarly, aloft currents may cause changes at aloft levels that are not felt at the surface. It is generally true, however, that density fluctuations decrease with altitude; in other words, that a given percentage variation of surface density with respect to the surface standard is greater than the accompanying percentage variations of aloft densities with respect to the standards assumed for aloft levels.

Accurate determination of aloft densities requires actual measurement of temperature and barometric pressure aloft. However, the general scheme followed in ballistics is to assume a standard altitude-density relation for the computation of basic tables, and to make adjustments for such variations from the assumed standard relation as may be observed in connection with service problems. The standard altitude-density relation used in ballistics is based on the standard surface density δ , previously defined, and is assumed to have the form

$$H_y = 10^{-.00001372y} \quad (405)$$

or

$$\log H_y = -.00001372y$$

in which y represents the altitude in feet and H_y the ratio of air density at any altitude y to air density at the surface. It follows from the above that *retardation varies directly as H_y* . The values of $\log H_y$ are tabulated in Table II, *Range and Ballistic Tables, 1935*, against y as argument. This table is often referred to as the H -table, and H_y as the H -function.

It may be considered that δ , and H_y together define a sort of standard atmosphere on which ballistic tables are based.* It is important to remember that this standard atmosphere implies not only the standard altitude-density relation H_y , but also the standard surface density δ ; in other words, that H_y , as defined by (405), is not applicable directly to other than standard surface density. This follows naturally from the conclusion already stated, viz., that density variations at the surface generally are greater than their accompanying variations aloft. This situation evidently implies that the entire altitude-density relation varies according to the surface density. It is possible, however, to make an adjustment in the observed non-standard surface density itself, so that this adjusted density, when used with the standard altitude-density relation H_y , may produce the equivalent of the observed density used with an altitude-density relation especially applica-

* This varies somewhat from the standard atmosphere as accepted by the Bureau of Standards, Weather Bureau, and some other activities, (for which see Table 695, *Smithsonian Physical Tables*, 8th Edition). The form assumed for H_y is particularly advantageous in connection with certain operations that are applied in the preparation and use of ballistic tables, as will be noted in article 630. The altitude-density relation defined by H_y , as given above, is applicable to the limits of altitude ordinarily reached by trajectories, but not beyond the limits of the troposphere (about 40,000 feet). For a more complete discussion of the altitude-density relation in connection with ballistics, including its application to levels within the stratosphere, see pp. 172-176, *Introduction to Ballistics* (U. S. Army, 1921), also pp. XVI-XVIII, *Exterior Ballistic Tables Based on Numerical Integration*, Volume I (U. S. Army, 1924)

ble to the latter. This course is much the simpler; among other things, it permits the adjustment both for non-standard surface density and for the non-standard altitude-density relation incident thereto, to be made by means of a change in the surface density factor δ alone, rather than by changes both in the latter and in H_v . The manner in which this is done requires a brief digression on methods of solution of the trajectory which are to be taken up in detail in Chapters 6 and 8.

421. The modern method of computing a trajectory is a step-by-step process which determines points in the trajectory in succession from the origin. The computation for each point is based on the correct physical data applicable to that point, including the air density. Since computations for tables are always based on standard conditions, this means that the density at each point is determined on the basis of standard surface density ($\delta=1$) and of the value of H_v for the ordinate of the given point, that is, on δH_v , with δ equal to unity. Now if a trajectory were to be computed for other than standard conditions by a similar process, we would use at each point the non-standard δ multiplied by an appropriate, non-standard H_v . We can, however, just as well consider H_v to remain at standard and the necessary change in the product δH_v to be absorbed by δ alone. This results in an artificial density factor which is, in fact, a percentage of the standard combination of δH_v , and which varies from point to point. Let us suppose now that a mean is taken of these varying density factors, and substituted in place of the actual surface density factor assumed for our computation. This mean value of δ is then a purely fictitious density factor, artificially set up to produce the equivalent, over the trajectory as a whole, of the use of the actual non-standard surface density and of the actual non-standard altitude-density relation applicable to the latter. The density factor determined according to the principles just outlined, is called the *ballistic density factor*, and we shall denote it by the symbol δ_b ; it is commonly referred to simply as the *ballistic density*.*

422. Since direct application of the above process obviously is impracticable for service problems, the following steps have been taken to reduce the process to a practical basis for the determination of ballistic density. For a wide variety of trajectories, the effects of given variations in density considered to apply only within restricted limits of the trajectory, have been compared to the effects of like variations considered to apply to the entire trajectory. From these comparisons *air-density weighting factors* have been deduced for portions of the trajectory lying within various zones of altitude, so that the effects of the several density factors which pertain to the several zones of the trajectory may be duly weighted and combined to give a weighted mean density factor, or *ballistic density*, for the entire trajectory. For example, the air-density weighting factors for a trajectory having a maximum ordinate of 3000 feet are .20 for the zone 0-600 feet, .28 for the zone 600-1500 feet, and .52 for the zone 1500-3000 feet. Now if δ_1 , δ_2 , and δ_3 , respectively, are the density factors for these zones, then $.20\delta_1 + .28\delta_2 + .52\delta_3$ is the ballistic density for the trajectory. It is to be understood that the ballistic density accounts for variations from the assumed standard surface density (δ_s) and from the assumed

* More properly, the ballistic density is the actual density (i.e., weight per unit volume) to which the ballistic density factor corresponds. However, since it is always the density factor, and not the density itself, that is used in practical applications, it is common practice to use the terms "density," "surface density," "ballistic density," etc. when referring, in fact, to the corresponding factors.

standard altitude-density relation (H_s); in other words, it states the percentage variation of the actual atmosphere in a given case from the standard atmosphere, within the limits of the given trajectory.

423. The ballistic density for a given case may be found by either of two methods, viz.,

- (a) By actually *measuring* the densities in various zones aloft, and applying the air-density weighting factors to the ratios of these observed densities to the standard densities for their respective zones.
- (b) By *assuming* densities for various zones aloft, on the basis only of the measured surface density, and using these, as in (a) above, in place of values actually observed aloft.

Determination
of ballistic
density from
aloft obser-
vations

The first of the above methods is the only one which accounts for conditions as they actually exist, and which allows for unpredictable variations in the atmosphere. It is used when aloft observations are available.* Such observations can be obtained only by specially equipped airplanes, and hence often are not

available.

The second of the above methods is designed to provide for an approximate determination of the ballistic density when facilities for obtaining aloft observations are not at hand, or when conditions are such as to render aerological flights impracticable. It depends upon an assumption of aloft densities under various conditions of surface density, based on a study of a great many observations taken in the past.† In this method only the surface density is determined by actual observation, and the aloft densities corresponding thereto are assumed to follow the average relation based on past experience, as noted above. In order to simplify matters, tables have been prepared which embody these average relations, and from which the ballistic density corresponding to any surface density and maximum ordinate may be found directly. Extracts from these tables are given in Table IV, *Range and Ballistic Tables, 1935*.

Determination
of ballistic
density from
surface
observations

424. The ballistic density for a given trajectory is found from Table IV by entering the latter with the surface density factor as horizontal argument and the maximum ordinate of the trajectory as vertical argument. This requires that the surface density factor, i.e., δ , first be found from Table III, with the surface temperature and pressure as arguments. An illustration will make this clear.

Given: Surface temperature, 84°F. ; surface barometric pressure, 29.90 inches.

Find: The ballistic density for a trajectory having a maximum ordinate of 18,000 feet.

From Table III, with the arguments 84° and 29.90 inches, we find

$$\delta = .960.$$

From Table IV, with the arguments $\delta = .960$ and $y_s = 18,000$ feet, we find

$$\delta_b = .993.$$

* At the present time the observed ballistic density for various maximum ordinates is determined by aerological units attached to force flagships, and transmitted to vessels in the vicinity of the latter. A complete description of the details involved in this determination is given in *Technical Regulations No. 1236-1* (U. S. War Department, 1934).

† Ref. *Aeronautical Meteorology*, Gregg; also *Monthly Weather Review, Supplement No. 20* (U. S. Weather Bureau publication no. 768, May, 1922).

This example shows that although in this case the variation from standard density is 4.0% at the surface, the weighted mean variation for the entire trajectory is only 0.7%, or about one-sixth of that indicated at the surface. Inspection of other values in the table indicates that the range of variations in aloft densities is materially less than that in the corresponding surface densities, which is in accord with the conclusions previously stated. For example, for values of δ which vary from .900 to 1.100, the corresponding values of δ_0 , for a trajectory having a maximum ordinate of 18,000 feet, vary only from .967 to 1.054.*

425. It seems appropriate at this point to make a general appraisal of what may reasonably be expected, in the way of accuracy, from the whole scheme for taking into account variations in air density, as outlined in the foregoing articles. First of all, even when aloft densities actually are measured, they are combined into a weighted mean by the application of weighting factors which are designed to serve for a wide variety of trajectories and which must, therefore, constitute a rather broad average. Considering, however, the degree of approximation that must be accepted in the basic density observations themselves, under practical service conditions, the use of anything more elaborate than the average weighting factors hardly is justifiable. Table IV involves not only the average weighting factors, but also a broad average of observations designed to apply without regard to time of day, locality, climate, season, or numerous other factors that may influence the altitude-density relation. Table IV therefore can be regarded only as a substitute for the better determinations which are based on actual observations.

It may be accepted that any errors incident to the approximations entering into the determination of a ballistic density from *actual aloft observations*, as outlined above, will be of a small order in comparison with other errors of gunfire. The order of error involved in the use of Table IV, although it may materially exceed that of the observational method, nevertheless has every likelihood of being much smaller than that incident to the use only of the surface density factor.†

426. Atmospheric density varies only slightly with humidity, and the effect of such variations from the standard humidity as are likely to occur in practice may be disregarded altogether. Variations in atmospheric density due to variations in humidity, within the probable extremes of the latter over sea areas, are limited to about $(\pm) 0.3\%$,‡ with a corresponding, limiting effect of only about 0.1% on the range of a trajectory.

427. The temperature of the air, entirely apart from its effect on atmospheric density, affects the elasticity of the air. The latter, in turn, affects the

* Under the assumptions made in arriving at the conception of a ballistic density, the latter, as determined from these tables, would be expected to be equal to unity at all altitudes when the surface density is at standard (i.e., $\delta = 1$). That it does not actually fulfill these conditions is due to the fact that the standard altitude-density relation (H_s) assumed to apply to standard surface density ($\delta = 1$), does not agree exactly with the corresponding observed relation which is embodied in Table IV. The slight variation of H_s as assumed for ballistic tables, from the observed average altitude-density relation for $\delta = 1$, can be accounted for by employing the ballistic density corresponding to $\delta = 1$ in connection with values from the ballistic tables.

† See also art. 1018.

‡ A table showing the effect on atmospheric density of variations in humidity, is given in Table VIII, *Technical Regulations No. 1936-1* (U. S. War Department, 1934).

The
temperature-
elasticity
effect

velocity of wave propagation in air (i.e., the velocity of sound).

The dependence of the retardation function on the velocity of sound already has been noted in articles 409 and 411. A change in the velocity of sound causes a change in the retardation function throughout its entire length, but the effect of this change is much greater in the critical portion in the vicinity of the velocity of sound than elsewhere. This may be seen by examining the graph of the J -function (Figure 9). There is every reason to believe that the abrupt change in this curve is closely associated with the velocity of sound, and hence that the point where this abrupt change occurs will shift as the velocity of sound shifts. At the steep portion of the curve just preceding the velocity of sound, even a small shift has an appreciable effect on the relation between retardation and velocity, although elsewhere the effect of a similar shift is much less.

It will be apparent that a correction for the effect of a variation in elasticity due to a change in temperature, must be treated as a correction to the retardation function itself, and hence it takes on all the complications of the latter. Fortunately the effect of variations in elasticity, within the limits of such variations and of the velocities normally encountered in practice, is small, and no serious error results from neglecting it.* For high-velocity guns the amount of this effect on the entire trajectory, measured in terms of the percentage variation in total range resulting therefrom, does not exceed about .02% per degree (F.) of variation from the standard temperature (59°). Through its operation on the air density, the effect of the same one degree of variation in temperature may be as great as about seven times the above.

428. For convenience in notation we will now combine the surface density factor and the several factors pertaining to the projectile as follows.

The
ballistic
coefficient C

$$C = \frac{w}{\delta d^2} \quad (406)$$

C is called the *ballistic coefficient*, and it is a measure of comparison between the retardation of a given projectile in air of a given surface density, and the retardation of the standard projectile in air of standard surface density. According to the relation stated above, *retardation varies inversely as the ballistic coefficient, C .*

We may now consider the values of G_* in the G -table to express retardation for the conditions under which $C = 1$, and for any given set of conditions the retardation, according to the G -function, is defined by

$$R_* = \frac{G_*}{C} \quad (407)$$

For example, to find the retardation at 3000 f.s. for a 5" projectile whose weight is 50 lbs. and coefficient of form .600, in air for which $\delta = .900$, we have

* Consideration of the temperature-elasticity effect has entered the field of exterior ballistics only comparatively recently, and subsequently to the determination of the G -function. The latter therefore takes no account of this effect and retardations tabulated in the G -table necessarily represent an average elasticity corresponding to the range of temperatures under which the Gåvre firings were conducted. As the normal average of temperatures encountered in practice probably varies but little from the Gåvre average, any error resulting from omission of a temperature-elasticity correction for variation of surface temperature from the assumed standard (59°F.) should remain small. For trajectories which ascend to great altitudes and hence encounter materially lower temperatures aloft, such a correction becomes of greater significance, and the most recent ballistic tables include it.

$$C = \frac{50}{.900 \times .600 \times 25} = 3.7037.$$

In Table I, at 3000 f.s., we find $G_v = 973.0$ f.s.s., whence the retardation for the assumed case is

$$R_a = \frac{973.0}{3.7037} = 262.71 \text{ f.s.s.}$$

429. Range tables are always based on standard air density at the surface (i.e., $\delta = 1$) and on the standard altitude-density relation defined by H_v . The special symbol E is used to define retardation under these conditions; that is, E denotes the retardation of a given projectile due to the resistance of air of standard density at the surface but corrected for variation of density with altitude, whence

$$E = \frac{G_v \times H_v}{C} \quad (408)$$

in which C is always based on $\delta = 1$. For example, in making a range-table computation for a 5" gun whose projectile weighs 50 lbs. and has a coefficient of form equal to .600, the retardation at a point in the trajectory where the remaining velocity is 2000 f.s. ($v = 2000$) and the altitude is 1000 feet ($y = 1000$), is found as follows. From Table I we find $G_v = 510.5$ f.s.s. From Table II we find $\log H_v = 9.98628 - 10$. With $\delta = 1$ we compute $C = 3.3333$.

$G_v = 510.5$	$\log 2.70800$
$H_v =$	$\log 9.98628 - 10$
$C = 3.3333$	$\text{colog } 9.47712 - 10$
$E = 148.39 \text{ f.s.s.}$	$\log 2.17140$

430. The following is an elementary example of the process of finding the coefficient of form.

Given: A 6" projectile weighing 105 lbs. was fired practically horizontally through two pairs of screens. The distance from the gun to the midpoint of the first pair was 400 feet, and to the midpoint of the second pair 915 feet. The mean of ten shots gave as the mean velocity between the first pair 2211.7 f.s. and between the second pair 2182.3 f.s. The temperature of the air was 57°F. and the barometer 29.75 inches.

Find: The coefficient of form of this projectile according to the G -function.

Elementary
determination
of i

We may proceed to determine the retardation from the experimental data exactly as was done in article 404.

$(v_1 - v_2) = 29.4$	$\log 1.46835$
$(v_1 + v_2) = 4394$	$\log 3.64286$
$2l = 1030$	$\log 3.01284$
$R_a = 125.42 \text{ f.s.s.}$	$\text{colog } 6.98716 - 10$
	$\log 2.09837$

The actual retardation was therefore 125.42 f.s.s. at the velocity 2197 f.s. (approximately). According to the G -function the retardation of the standard projectile at 2197 f.s. is $G_v = 591.7$ f.s.s. The disagreement between the experimentally measured value and the tabular value must be accounted for by the ballistic co-

efficient, and since in the latter we can directly evaluate δ , w , and d^2 , any remaining disagreement must be assigned to i . We will therefore expand (407) and then rewrite it in terms of i , as follows

$$R_a = G_v \times \frac{\delta i d^2}{w}$$

$$i = \frac{R_a}{G_v} \times \frac{w}{\delta d^2} \quad (409)$$

We may now solve (409) to find the value of i that is required to establish the required agreement between the measured retardation and the G -function. From Table III we find $\delta = 1.012$.

$R_a = 125.42$	log 2.09837
$\delta = 1.012$	log 0.00518....colog 9.99482-10
$w = 105$	log 2.02119
$G_v = 591.7$	log 2.77210....colog 7.22790-10
$d^2 = 36$	log 1.55630....colog 8.44370-10
$i = .61092$	log 9.78598-10

The coefficient of form we have thus found, practically .61, has only a limited application. For other trajectories than the one here involved it may differ somewhat, for reasons that have already been gone into at length (arts. 413-416). But even a very elementary determination of this type may be useful for a first approximation of the coefficient of form of a projectile.

EXERCISES

1. Compute the values of $\log C$ for the cases listed in the following table.

Problem	DATA					ANSWERS
	Projectile			Atmosphere (surface)		log C
	d (in.)	w (lbs.)	i	Temp. (°F.)	Bar. (in.)	
1.....	3	13	1.00	61	29.80	0.15710
2.....	4	33	0.67	65	29.60	0.49268
3.....	5	50	0.59	57	30.25	0.51784
4.....	6	105	0.61	70	30.50	0.67524
5.....	8	260	0.61	85	29.75	0.84437
6.....	12	870	0.61	93	30.20	1.01811
7.....	14	1400	0.70	69	29.80	1.01401
8.....	16	2100	0.61	32	30.15	1.09482

2. Find the surface density factor δ (Table III), and the ballistic density δ_b (Table IV), for the following cases.

Problem	DATA			ANSWERS	
	Atmosphere (surface)		Maximum ordinate (feet)	δ	δ_b
	Temp. (°F.)	Bar. (in.)			
1.....	65	29.60	1000	.990	.991
2.....	85	29.75	18000	.953	.991
3.....	57	30.25	8000	1.030	1.026
4.....	69	29.80	13000	.988	1.003
5.....	32	30.15	15000	1.081	1.050

3. Given the measured velocities of a projectile at two points, as determined by firing horizontally through screens, compute the retardation and resistance due to the atmosphere, and state the velocity (approximately) to which the results apply ($g = 32.16$).

Problem	DATA				ANSWERS		
	w (lbs.)	Distance between points of measure- ment (ft.)	Measured velocities at		R_a (f.s.s.)	R_f (lbs.)	Apply to velocity (f.s.)
			First point (f.s.)	Second point (f.s.)			
1.....	13	500	2680	2572	567.20	229.28	2626
2.....	50	500	3140	3088	323.85	503.50	3114
3.....	870	500	2870	2854	91.58	2477.5	2862
4.....	2100	500	2590	2578	62.02	4049.4	2584

4. Find the retardation of a projectile at the earth's surface, according to the G -function, for the following cases.

Problem	DATA						ANSWERS
	Projectile			Atmosphere (surface)		Velocity (f.s.)	R_a (f.s.s.)
	d (in.)	w (lbs.)	i	Temp. (°F.)	Bar. (in.)		
1.....	4	33	0.67	65	29.60	2890	293.82
2.....	6	105	0.61	70	30.50	2584	180.92
3.....	8	260	0.61	85	29.75	2730	119.06
4.....	14	1400	0.70	69	29.80	2580	73.58
5.....	16	2100	0.61	32	30.15	2565	60.52

5. Given the measured velocities of a projectile at two points, as determined by firing horizontally through screens, compute the coefficient of form according to the G -function. (Use V to nearest f.s. in entering G -table.)

Problem	DATA							ANSWERS
	Projectile		Atmosphere (surface)		Measured velocities at		Distance between points of measure- ment (ft.)	i
	d (in.)	w (lbs.)	Temp. (°F)	Bar. (in.)	First point (f.s.)	Second point (f.s.)		
1.....	3	13	85	29.75	2660	2562	500	1.0015
2.....	5	50	32	30.15	3120	3066	500	.60272
3.....	12	870	57	30.25	2840	2824	500	.60187
4.....	16	2100	69	29.80	2610	2599	500	.61687

CHAPTER 5

THE DIFFERENTIAL EQUATIONS OF MOTION OF A PROJECTILE IN AIR AND SIACCI'S METHOD OF SOLVING THEM.

New Symbols Introduced

ρ	Radius of curvature of the trajectory at any point.
ds	Differential of the length of arc of the trajectory.
$f(v)$	A general notation for the function of velocity taken to express the retardation of a projectile due to air resistance, under the condition $C = 1$.
u	The pseudo velocity; a component of the remaining velocity obtained by projecting the latter vertically upon the line of departure or a line parallel thereto.
f_a	The altitude factor in Siacci's Method; the ratio of air density at the surface to air density at the height of the mean ordinate of the trajectory.
β	An approximate constant devised by Siacci to reduce the differential equations to an integrable form.
C_a	The Siacci C ; the ballistic coefficient augmented by Siacci's approximate mean-value constants f_a and β , and used only with Siacci's Method.
S_u, T_u, I_u, A_u	These represent integral expressions occurring in Siacci's Method, which have been tabulated.

501. The motion of a projectile in air, imparted to it initially by the gun, is modified in flight by two forces, namely, the force of gravity and the force of air resistance. In terms of their corresponding accelerations, and according to the

general law $F = \frac{w}{g} \alpha$, these forces are represented by the expressions, $\frac{w}{g} \times g$,

or w , for the force of gravity, and $\frac{w}{g} \times R_a$ for the force of air resistance. The

Forces acting upon the projectile trajectory being curved, as a consequence of the action of gravity, a centrifugal force is set up; the latter, for any point in the trajectory where the radius of curvature is ρ and the remaining velocity v , is rep-

resented by the expression $\frac{w}{g} \times \frac{v^2}{\rho}$. The force of gravity operates vertically and

the force of air resistance in the direction of the tangent to the trajectory, and the direction of the centrifugal force is normal to the latter.

* Ref. any standard work on mechanics.

At any point P in the trajectory (Figure 11) the force of air resistance has a zero component in the direction normal to the tangent at that point, i.e., along the radius of curvature OP at that point. The angle of inclination at the given point being θ , the force of gravity has in this same direction the component $w \cos \theta$ acting directly toward the center of curvature, and the latter is exactly balanced by the centrifugal force acting directly away from the center of curvature. We have, then,

$$\begin{aligned} \frac{w}{g} \times \frac{v^2}{\rho} &= w \cos \theta \\ \frac{v^2}{\rho} &= g \cos \theta. \end{aligned} \quad (501)$$

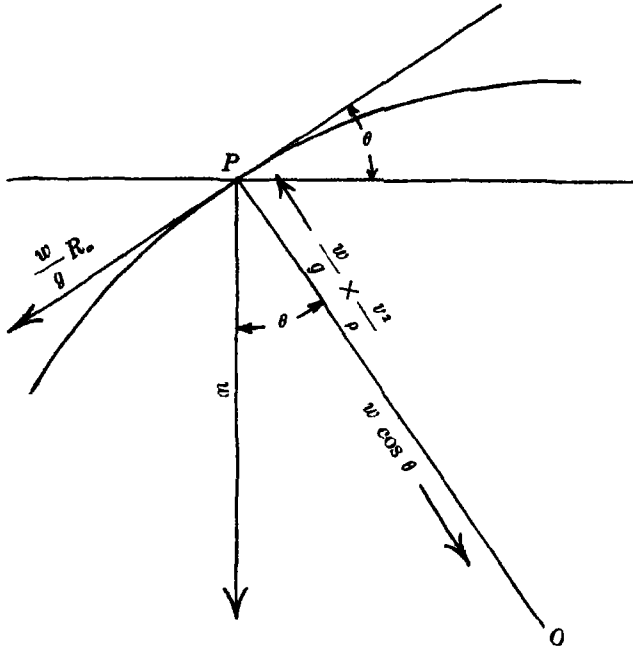


FIGURE 11

In terms of s , denoting length measured along the arc of the trajectory, the radius of curvature of the trajectory at point P is $\rho = -\frac{ds^2}{d\theta}$; substituting this value in

(501), and transposing, we have

$$ds \cos \theta = -\frac{v^2}{g} d\theta.$$

* Ref. any standard work on mechanics.

Since $ds \cos \theta = dx$, we may write from the above

$$dx = - \frac{v^2}{g} d\theta \quad (502)$$

Also, since $dy = dx \tan \theta$, we may write from (502)

$$dy = - \frac{v^2}{g} \tan \theta d\theta. \quad (503)$$

Dividing both sides of (502) by dt

$$\frac{dx}{dt} = - \frac{v^2}{g} \frac{d\theta}{dt}$$

and since $\frac{dx}{dt} = v \cos \theta$, we have

$$v \cos \theta = - \frac{v^2}{g} \frac{d\theta}{dt}$$

whence

$$dt = - \frac{v}{g} \sec \theta d\theta. \quad (504)$$

502. We have already found that the retardation of a projectile due to air resistance is not expressible in any simple form, but we know that it is a function of the velocity. Mayevski's functions state this in a formal manner by means of the expression (403), while the Gâvre function states it in tabular fashion. For convenience, we shall use the notation $f(v)$ (i.e., a function of the velocity) to express the retardation due to air resistance for the condition $C=1$. We also have found that the retardation due to air resistance is inversely proportional to the ballistic coefficient, which combines the various constants that pertain to the projectile and atmospheric conditions of any particular case. Acceleration is denoted by $\frac{dv}{dt}$, and we may therefore write

$$\frac{dv}{dt} = - \frac{1}{C} f(v).$$

Also, since $v \cos \theta$ denotes the horizontal velocity we may write for the horizontal component of acceleration

$$\frac{d(v \cos \theta)}{dt} = - \frac{1}{C} f(v) \cos \theta.$$

Substituting for dt in the above its value as found from (504), we have

$$\frac{g \, d(v \cos \theta)}{-v \sec \theta \, d\theta} = - \frac{1}{C} f(v) \cos \theta$$

$$d(v \cos \theta) = \frac{1}{C} \frac{v}{g} f(v) d\theta. \quad (505)$$

503. We now have derived the following differential equations, which state the relations among several elements of the trajectory in air.

$$dx = - \frac{v^2}{g} d\theta \quad (502)$$

Differential
equations
pertaining
to the
trajectory
in air

$$dy = - \frac{v^3}{g} \tan \theta d\theta \quad (503)$$

$$dt = - \frac{v}{g} \sec \theta d\theta \quad (504)$$

$$d(v \cos \theta) = \frac{1}{C} \frac{v}{g} f(v) d\theta. \quad (505)$$

Each of the equations (502), (503), and (504) contains the variables v and θ , in addition to the variables to be solved for (i.e., dx , dy , dt). Equation (505) gives a relation between v and θ , and the solution of the entire system represented by the four equations depends upon the solution of (505). We shall therefore refer to (505) as the *chief equation*;* for if values of v corresponding to any given values of θ

are available, then (502), (503), and (504) can be integrated within any limits of θ , and the values of x , y , and t for any point in the trajectory may be determined. The solution of the chief equation, however, depends in turn upon knowledge of the retardation function, represented by $f(v)$, which appears in it. We have already seen that $f(v)$ itself is a very complex function, and hence many difficulties are involved in the solution of the chief equation.

504. Historically, the methods of solution of the chief equation went through stages closely related to those we have already noted in the development of the retardation function (art. 409). The earliest methods concerned themselves with solutions under monomial laws of retardation, that is, laws assuming retardation to vary as the square, cube, fourth power, etc., of the velocity. As early as 1719, Bernoulli stated the integrations in terms of a law assuming retardation to vary as the n th power of velocity, that is, for $f(v) = kv^n$ (n being an integer). In 1744, d'Alembert gave a solution for a more general law of the form $f(v) = kv^n + q$. A further mode of attack, designed to establish laws which would more accurately represent the known velocity-retardation relation and yet leave it possible to apply formal integration to the chief equation, lay in deriving from the latter numerous forms which would serve the purposes of integration, and in attempting

Exact solution
of the chief
equation in
terms of an
approximate
retardation law

to fit the velocity-retardation relation into one of them. Numerous such forms were offered from time to time, Siacci alone having given out fourteen of them, but no advantages were derived from this mode of attack. The significant feature of all of these early methods was the attempt to reduce the retardation function to a form permitting formal integration of the chief equation in a single

* If values of v and their corresponding values of θ be laid off as vectors from a common origin, with v as the vector length and θ as its inclination with respect to the axis of coordinates, the curve connecting the extremities of these vectors is called, in mechanics, the *hodograph* of the motion thus represented. Equation (505) is the equation of the hodograph for the motion of a projectile in air.

step; that is, these methods all resorted to approximate forms of the retardation function, although adhering otherwise to exact solutions of the chief equation.

505. The general acceptance of the system of establishing zones of velocity, with a separate monomial law for each zone, was followed, naturally, by methods of solution appropriate to this scheme. As has already been noted (art. 409), the most generally favored of the systems based on zones was Mayevski's series,

which is represented by the general expression $R_z = \frac{A}{C} v^a$, with A and a varying

from zone to zone but remaining constant within a given zone. To apply this expression to the chief equation we replace $f(v)$ with Av^a and have

The chief equation
in terms of
Mayevski's functions

$$d(v \cos \theta) = \frac{1}{C} \frac{A}{g} v^{(a+1)} d\theta. \quad (506)$$

Considering that both A and a vary from zone to zone it is necessary, first of all, to perform the integration of (506) in zones. Moreover, in order to separate the variables it is necessary to resort to some form of approximation.

Various methods of solving (506), or forms of similar nature (i.e., based on zones), have been offered. They are all characterized by the establishment, among the variables of the chief equation itself, of certain relations which remain nearly constant for a given trajectory and whose approximate mean value for a given trajectory may be evaluated independently of the integration. The separation of these relations from the variables of the chief equation, in the form of mean-value constants, then leaves the equation integrable. However, these devices do not eliminate the necessity of performing the integration zone by zone, with the appropriate retardation function applied in each zone. Several of these methods have been developed to the extent of preparing tables of the integrated values of one or more forms employed in the final evaluation of the differential equations, and thus have eliminated the necessity of actually performing integrations zone by zone for each problem.*

It is noteworthy that the type of procedure just discussed accepts approximation in connection with the form of the chief equation, but adheres without compromise to the best experimental determination of the retardation function that may be available. This is just the reverse of the earlier mode of attack that has been discussed in the foregoing article. Also, it is significant that the later procedure abandons the attempt to perform the necessary integrations for an entire trajectory in a single step, and resorts to separate integrations with respect to each of the several velocity zones that may be included within the limits of a given trajectory.

Siacci's Method

506. As an illustration of methods of the type mentioned in the foregoing article we shall examine briefly *Siacci's Method*. Col. Siacci, an Italian artillery officer, published his method initially in 1880, and in 1888 and 1896 he published extensions of his method, including additional refinements and tables based on different retardation functions than first assumed. His method, while employing devices similar in character to those of several contemporary methods, achieved

* An exhaustive discussion of methods of the type noted here, as well as of the earlier types mentioned in article 504, is given in pp. 88-215, *Handbook of Ballistics*, Vol. I, by Cram and Becker.

greater simplicity than did the other methods and was widely used by the ballisticians of nearly all countries, including the United States, until about the time of the World War. In the United States, Siacci's Method was further simplified in its application to actual solutions of the trajectory, although not altered in principle, by Col. Ingalls (U.S. Army). Siacci adapted his method not only to Mayevski's retardation functions but also to other retardation functions of the same type. In the United States, Mayevski's functions in the form given by Ingalls (art. 409) are used with Siacci's Method.

507. One of the approximations made by Siacci was designed to avoid the complications that would result from introducing into the chief equation the function representing the change of air density with altitude (that is, a function such as H_z , which already has been defined in article 420). It consisted simply of assuming that this function can be reduced to a mean value for an entire

Approximation of
the altitude-
density relation.
The altitude
factor f_a .

trajectory, and in this manner be treated as a constant. Various procedures have been offered for determining such a mean air density; the practice in our services, in connection with Siacci's Method, has been to assume the mean density for the entire trajectory to be the density at the height of its mean ordinate, and to take the latter as equal to two-thirds of the height of the maximum ordinate (which is exact only for a true parabola). This approximation is found to be satisfactory for trajectories having angles of departure up to about 15° .* In our services this altitude-density relation has been denoted by the symbol f_a and called the altitude factor. The factor f_a has been chosen to represent the ratio of air density at the surface to air density at the height of the mean ordinate, and hence it operates as a divisor of the surface density factor δ , or as a multiplier in C (note that the ratio H_z has been taken in the reverse sense).

Since the value of f_a depends upon the maximum ordinate, and the determination of the latter, as well as of all other elements of a trajectory, depends upon prior knowledge of f_a , it is necessary in the Siacci Method to make a solution first without f_a and to determine an approximate value of this factor from the approximate maximum ordinate thus obtained. By means of further successive approximations the value of f_a may then be established with any desired degree of accuracy and the solution completed.

508. The principal feature of Siacci's Method is the device leading to the separation of the variables of the chief equation, for the purpose of putting the

latter in shape for formal integration. This device consists of introducing into the chief equation a relation between the remaining velocity v and a component u of the latter obtained by projecting v vertically upon the line of departure (or a line parallel thereto).

This component u is called the *pseudo velocity*. As shown by Figure 12, this relation is

$$u \cos \phi = v \cos \theta \quad (507)$$

also

$$v = \frac{u \cos \phi}{\cos \theta} \quad (508)$$

* A somewhat closer approximation for the mean ordinate is given on page VIII, *Artillery Circular M* (1917); it shows that the assumption $y_m = \frac{1}{2}y_0$ is in error by less than 1% for ordinary trajectories having angles of departure less than about 15° .

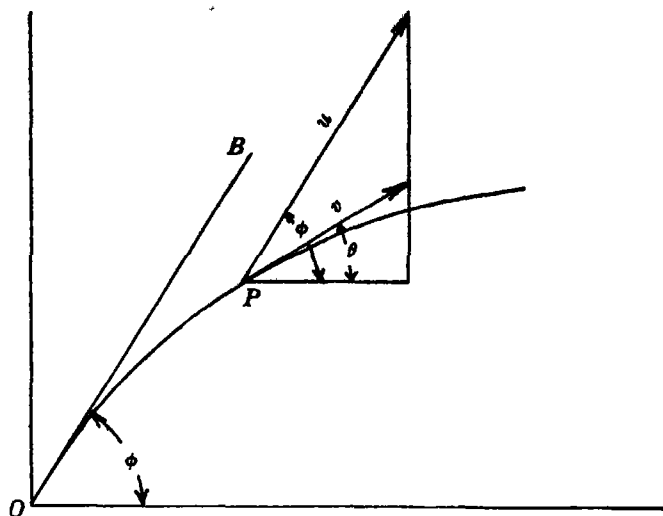


FIGURE 12

Substituting these equivalents of $v \cos \theta$ and of v in the left-hand and right-hand members of (506), and introducing into the latter at the same time also the altitude factor f_a , we have

$$d(u \cos \phi) = \frac{1}{f_a C} \times \frac{A}{g} \left(\frac{u \cos \phi}{\cos \theta} \right)^{(a+1)} d\theta.$$

Writing $du \cos \phi$ for $d(u \cos \phi)$ in the above, and expanding the term in the bracket, we can transpose the equation to the form

$$f_a C \times \frac{g}{A} \times \frac{du}{u^{(a+1)}} = \frac{\cos^a \phi}{\cos^{(a+1)} \theta} d\theta. \quad (509)$$

The following operations are now applied to terms in the right-hand member of (509),

$$\begin{aligned} \cos^a \phi &= \cos^{(a-2)} \phi \cos^2 \phi \\ \frac{1}{\cos^{(a+1)} \theta} &= \frac{1}{\cos^{(a-1)} \theta} \sec^2 \theta \end{aligned}$$

and substituting these equivalents in (509) we have

$$f_a C \times \frac{g}{A} \times \frac{du}{u^{(a+1)}} = \left(\frac{\cos^{(a-2)} \phi}{\cos^{(a-1)} \theta} \right) \cos^2 \phi \sec^2 \theta d\theta. \quad (510)$$

509. Equation (510) has been put into the form shown above deliberately for the purpose of creating therein the expression $\left(\frac{\cos^{(a-2)} \phi}{\cos^{(a-1)} \theta} \right)$, since it has been

found that a mean value of this expression for an entire trajectory can be approximated satisfactorily in terms of initial elements, and hence can be determined in-

independently of the variables θ and u and hence of the integration of (510). We will therefore denote this expression by the special symbol β , so that

$$\text{The factor } \beta \qquad \beta = \frac{\cos^{(a-2)} \phi}{\cos^{(a-1)} \theta} \qquad (511)$$

and treat β as being independent of the variables of (510), i.e., as a constant.

We may now proceed to put (510) into form for integration, first replacing the value in the bracket by its symbol β , whence

$$f_a C \times \frac{g}{A} \times \frac{du}{u^{(a+1)}} = \beta \cos^2 \phi \sec^2 \theta d\theta$$

and we may then transpose the above to the form

$$\sec^2 \theta d\theta = \left(\frac{f_a C}{\beta} \times \frac{g}{A \cos^2 \phi} \right) \frac{du}{u^{(a+1)}}. \qquad (512)$$

Since f_a and β are to be regarded as constants applying to an entire trajectory, we may simplify our notation by combining them with C , which already contains other such constants; thus

$$\text{The Siacci } C, \text{ or } C_s \qquad C_s = \frac{f_a}{\beta} C. \qquad (513)$$

This artificial combination represented by C_s is called the *Siacci C*, and it is a special form of the ballistic coefficient to be used with Siacci's Method only.

We may then finally write the chief equation, from (512), in the form

$$\text{The chief equation in terms of Mayevski's functions and Siacci } C \qquad \sec^2 \theta d\theta = \left(\frac{C_s g}{A \cos^2 \phi} \right) \frac{du}{u^{(a+1)}}. \qquad (514)$$

Now all the terms in the bracket in the right-hand member of (514) are independent of the variables θ and u , and the latter are completely separated. The integration may then be expressed as follows,

$$\int_{\phi}^{\theta} \sec^2 \theta d\theta = \frac{C_s g}{A \cos^2 \phi} \int_V^u \frac{du}{u^{(a+1)}} \qquad (515)$$

the limits for θ being its value ϕ at the origin and its general value θ at any point in the trajectory, and for u its value V at the origin and its general value u at any point in the trajectory.

The integration of (515) presents no difficulties except that for the right-hand member it must be performed in as many steps as there are velocity zones within the chosen limits of velocity. For example, if Mayevski's retardation functions are to be used, and if the integration is to be between the velocities 2900 f.s. and 1600 f.s., then the integration must proceed through the following steps: from 2900 f.s. to 2600 f.s. with $a = 1.55$ and $\log A = 7.60905 - 10$; from 2600 f.s. to 1800 f.s. with $a = 1.7$ and $\log A = 7.09620 - 10$; and from 1800 f.s. to 1600 f.s. with $a = 2$ and $\log A = 6.11926 - 10$ (art. 409).

510. We will now examine very briefly the manner of applying the solution of the chief equation to the solution of another of the differential equations (art. 503). The equation for range is

$$dx = -\frac{v^2}{g} d\theta. \quad (502)$$

From (508) we have the relation $v = u \cos \phi \sec \theta$; substituting this in (502) we have

$$dx = -\left(\frac{u^2 \cos^2 \phi}{g}\right) \sec^2 \theta d\theta$$

and substituting in the above the value of $\sec^2 \theta d\theta$ as given by (514) we have

$$dx = -\left(\frac{u^2 \cos^2 \phi}{g}\right) \left(\frac{C_s g}{A \cos^2 \phi}\right) \frac{du}{u^{(a+1)}}$$

which simplifies to

$$dx = -\frac{C_s}{A} \times \frac{du}{u^{(a+1)}}. \quad (516)$$

The integration of (516), between the origin where $u = V$ and $x = 0$, and any point in the trajectory where velocity and range have the general values u and x , is then expressed as follows

Integration of the
differential equation
for range

$$\int_0^x dx = -\frac{C_s}{A} \int_V^u \frac{du}{u^{(a+1)}} \quad (517)$$

and this integration again offers no difficulties except that it must be performed in steps for each velocity zone within the limits of V and u , as already explained in the foregoing article.

A simplification of the ultimate process of solution for range may be achieved by tabulating the integration of the right-hand member of (517) for the entire range of velocities likely to be encountered in practice. This is done as follows. We will choose for the tabulation the quantity

The space
function

$$S_u = -\frac{1}{A} \int \frac{du}{u^{(a+1)}} \quad (518)$$

and starting with 3600 f.s. perform the integration successively, for small intervals of velocity, down to the lowest velocity likely to be used, changing the values of A and a as we pass from zone to zone. Having once done this, the value of the integral for any velocity may be found directly from the table by entering with the given velocity, and the solution of (517) is simplified to

$$x = C_s(S_u - S_V). \quad (519)$$

511. The integral denoted by S_u is called the *space function* in Siacci's Method. In a similar manner an *inclination function* I_u may be separated from (515) and tabulated. Operations similar to those shown above for the range equation (502), lead to an *altitude function* A_u when applied to (503), and to a *time function* T_u when applied to (504). Tables of each of these functions were prepared by Siacci. Ingalls prepared tables of the same functions, in English units, for velocities from 3600 f.s. to 100 f.s., using Mayevski's retardation expressions in the form given in article 409. Ingalls' tables appeared soon after the first publication of Siacci's Method and have been used by our services, in connection with the latter, since that time.

Tabulation of
integral func-
tions for Siacci's
Method

512. As shown in article 509, the factor β was arbitrarily chosen to represent the relation

$$\beta = \frac{\cos^{(a-2)} \phi}{\cos^{(a-1)} \theta} \quad (511)$$

and then treated as a constant (contained in C_1) in the chief equation (515). Since the cosine of a small angle varies but little from unity, it is evident that β does not vary much from unity for small values of ϕ and θ . For example, let us examine the values of β for the trajectory of the 16''2600 f.s. gun at about 13,300 yards, for which the angle of departure is about 7° , the angle of fall about 9° , and the striking velocity about 1821 f.s. Since the velocities on this trajectory are all in the zone 2600-1800 f.s., the value $a=1.7$ applies at all points. We can then evaluate β for several points in the trajectory as follows. At the origin $\theta=\phi=7^\circ$, whence

$$\beta = \frac{\cos^{-.3} 7^\circ}{\cos^{.7} 7^\circ} = 1.008.$$

At the summit $\theta=0^\circ$, whence

$$\beta = \frac{\cos^{-.3} 7^\circ}{\cos^{.7} 0^\circ} = 1.002.$$

At the point in the descending branch where $\theta=-\phi$

$$\beta = \frac{\cos^{-.3} 7^\circ}{\cos^{.7} 7^\circ} = 1.008.$$

At the point of fall $\theta=\omega=9^\circ$, whence

$$\beta = \frac{\cos^{-.3} 7^\circ}{\cos^{.7} 9^\circ} = 1.011.$$

The flat portion of the trajectory on both sides of the vertex is relatively much longer than the more curved portions near the ends; the portion beyond the point in the descending branch where $\theta=-\phi$ is only a small fraction of the whole length. A weighted mean of the above values is about 1.005, and this represents as accurately as necessary the mean value of β for the above trajectory. As the trajectory becomes more curved the mean value of β increases; for an angle of departure of 15° in the above case it becomes about 1.018. Siacci prepared fairly elaborate approximations of β for a wide variety of cases. The practice in our services has been to use $\beta=\sqrt{\sec \phi}$, which is found to give a satisfactory approximation for trajectories having angles of departure up to about 15° .

The significant feature in the choice of the factor β is that its value may be approximated from the value of ϕ , which is known. The device which makes possible the establishment of this fortunate relation between ϕ and θ is the introduction of u as defined by (507).

513. Our present purpose is to investigate chiefly the character of Siacci's Method. The devices leading to the adoption of the approximate mean-value constants f_0 and β , the introduction of these into the chief equation in such a manner as to render the latter subject to formal integration, the general character of the integration remaining to be performed, and the simplification of the latter (by means of tables of S_u) in the case of the differential equation for range, have been examined. No new principles are involved in applying the process to the remaining differential equations; the choice of additional integral expressions for tabulation has been touched upon in article 511. It is not necessary for our present purpose to go farther into the details of setting up the formulas and tables or of

making solutions. However, since Siacci's Method is still used, to a limited extent, by the U. S. Navy, a more complete treatment of the details leading to final solutions of the trajectory by this method is given in Appendix A.*

Siacci's Method, receiving wide recognition almost immediately after its first publication, remained the standard among ballisticians of nearly all countries for some thirty-five years, until the time of the World War, when it fell quickly to a secondary position. The reason for this will be apparent from an examination of the limitations imposed by the approximations used in this method, which are characteristic of all similar methods.

514. The use of a mean altitude-density factor pertaining to an entire trajectory (such as f_a) has the serious defect that it compromises the accuracy of all intermediate points of the trajectory in favor of the terminal point.

Limitations of Siacci's Method By resorting to sufficient elaboration in determining a mean ordinate, a satisfactory value of f_a may be determined even for trajectories having the greatest angles of departure used in practice, but any value of f_a so determined can satisfy the conditions for only one point. If it is desired to fix accurately a number of points in a trajectory, as is the case in antiaircraft problems, it is necessary either to accept a diminished degree of accuracy for all points other than the one for which f_a represents the correct mean, or else to determine a new f_a for each point to be found. Since accuracy at all points is essential in the computation of antiaircraft trajectories, the latter course becomes necessary in the application of Siacci's Method to such trajectories.

The same situation exists in the case of Siacci's other mean-value constant, β . The difficulties in determining a satisfactory value of this factor, even in the solution for terminal elements of trajectories having great angles of departure, are greater than in the case of f_a . The determination of satisfactory values of β for intermediate points, as required for antiaircraft trajectories, leads to further complications. The fortunate relation that β very nearly equals $\sqrt{\sec \phi}$ ceases to be true not only for trajectories having angles of departure greater than about 15° but also for points other than the point of fall. In dealing with either of the latter cases it is necessary to resort to more elaborate approximations of the mean value of β .

The great favor accorded Siacci's Method rested solely upon the latter's simplicity. The development of long-range fire and antiaircraft fire, and the consequent need to deal with trajectories of great length and curvature, and with other than terminal elements, imposed on Siacci's Method such an elaborate superstructure of approximations that its feature of simplicity disappeared almost completely. The natural outcome of this was the abandonment of the entire system of approximations and the development of methods of unlimited application.

At the time of this writing Siacci's Method is still used by the U. S. Navy, in the computation of range tables for surface fire, for angles of departure up to 15° . It is anticipated, however, that in the near future the more recent method will be extended to the computation of our range tables for all angles of departure. Ballistic tables based on the latter method, which are at the time of this writing in the course of preparation by the U. S. Army, will reduce to very simple terms the immediate problem of constructing range tables.

* The complete derivation of all formulas of the Siacci Method and Ingalls' simplification thereof, and detailed demonstrations of their solution according to the latest practice in the U. S. Navy, are given in the 1926 and 1930 editions of this book. Ingalls' tables are given in the 1926 and 1930 editions of *Range and Ballistic Tables*.

CHAPTER 6

THE SOLUTION OF THE TRAJECTORY IN AIR BY THE NUMERICAL INTEGRATION METHOD.

New Symbols Introduced

x', y'	Horizontal and vertical components, respectively, of the remaining velocity v of the projectile at any point x, y .
x'', y''	Horizontal and vertical components, respectively, of the total retardation experienced by the projectile at any point x, y .
x_0, y_0', v_0, E_0 , etc. . .	} The subscripts denote the point to which the values pertain; for instance, x_0 is the value of x at the point where the time is zero, i.e., at the origin; $y_{1/4}'$ is the value of y' at the point which has been reached by the projectile after 1/4 second of flight; $v_{1/2}$ is the value of v at the 1 1/2-second point; E_1 is the value of E at the 1-second point; and so on.
$x_{1/4}, y_{1/4}', v_{1/4}, E_{1/4}$, etc. . .	
$x_{1/2}, y_{1/2}', v_{1/2}, E_{1/2}$, etc. . .	
x_m'', x_m', y_m'', y_m'	The mean values of the retardation or velocity components for the particular interval in which the quantities are used.

601. The limitations of Siacci's Method and of other similar methods of approximation were appreciated long before these methods finally were relegated to a secondary position, during the period of the World War. That this did not happen much sooner may be attributed entirely to the additional labor incident to more accurate and universally applicable methods, and to the fact that the defects inherent in the approximation methods did not reach sufficiently great proportions, in connection with the comparatively restricted problems encountered prior to the World War, to warrant such additional labor.

It has long been known that the most complex of equations can be solved by a step-by-step process, and such procedure has been applied in astronomical computations for some two centuries. Euler, in 1753, gave a solution of this character for the ballistic problem. As described by Cranz in his *Handbook of Ballistics*, Vol. 1,

"His method depends on the summation of dx, dy, dt, ds . He treats the trajectory as a polygon of an infinite number of straight arcs Δs , and thence makes up the finite expression for the corresponding projections, Δx and Δy , as well as for the corresponding time Δt ; he then sums up the $\Delta x, \Delta y, \Delta t$ to x, y, t . He assumes the quadratic law of air resistance."

This description also fits the modern method remarkably well. Tables based on Euler's principle, and designed to eliminate the necessity for performing the tedious step-by-step integrations for each problem, appeared soon after this principle was first announced. The most noteworthy development along these

lines was given out by Otto, in 1842. In 1873, Bashforth published a method corresponding to that of Euler but based on the cubic law of air resistance.

Graphical equivalents of the step-by-step method of solution also have appeared from time to time. In 1828, Poncelet and Didion gave out a method which consisted essentially of constructing the trajectory graphically by a progression of arcs struck with successively determined values of the radius of curvature (which amounts practically to a step-by-step graphical construction according to the analysis given in article 501 and illustrated in Figure 11). Cranz, in 1896, proposed a graphical process that is a near equivalent to one of the computational methods now in favor. His process was based on short intervals of time, and depended on locating successive points along the trajectory by projecting forward along the tangent for the mean velocity during the particular interval, and vertically downward for the effect of gravity during the interval. In 1909, Cranz introduced a procedure which consisted essentially of graphing the differential equations (art. 503) and of making the necessary summations of dx , dy , dt by means of a planimeter or integrator.*

602. It may then be said that the comparatively recent adoption of the step-by-step process as the ultimate in methods of solution of the ballistic problem, is a reversion to a principle that was among the very earliest to be applied to this problem,—although, since its reappearance in recent years, it has often been thought of and referred to as a new departure in the science of ballistics. In its modern form, the step-by-step solution of the trajectory is carried out by the process now generally known as *numerical integration*, differing from Euler's Method chiefly in that it is a higher development of the latter and in that more complete and more accurate retardation data are now available for use with it. Numerical integration has been applied extensively to the solution of differential equations since long before its recent reappearance in the field of ballistics, and the development of the process has resulted from its general applications rather than from its application in the latter field. Modern texts dealing with the solution of differential equations generally include treatments of the theory and practice of numerical integration.

Numerical integration methods for the solution of trajectories were developed independently and practically concurrently, both here and abroad, during the latter part of the World War. The French and British, however, led in the practical application of such methods and before the end of the war they were using ballistic tables based on numerical integration. In the United States the application of numerical integration to the solution of trajectories was due chiefly to F. R. Moulton, professor of astronomy at the University of Chicago. Moulton, having been commissioned a Major in the United States Army, was placed in charge of the Ballistics Branch of the Ordnance Department of that service early in 1918, and shortly thereafter gave out the essence of the methods which have since that time remained in favor in this country.†

603. Numerical integration consists essentially of evaluating a differential relation between two variables by assigning numerical values to one of them in

* For further description and discussion of the methods here mentioned, and of many others, see *Handbook of Ballistics*, Vol. I, Cranz and Becker.

† In his book, *New Methods in Exterior Ballistics*, published in 1926, Moulton gives an account of his investigations in the field of ballistics both during the war period and subsequently in connection with his post-graduate instruction of officers of the U. S. Army and U. S. Navy at the University of Chicago.

Nature of numerical integration successive, regular intervals, and finding the corresponding increments of the other for the chosen intervals. The integrated value of one of the variables with respect to any chosen limits of the other may be determined by summing the successive increments of the former within these limits. In applying numerical integration to the solution of the differential equations of the trajectory as they are stated in article 503, we are still confronted with the necessity of solving first the chief equation (505). Since the latter expresses a relation between v and θ , we can either integrate v by assuming successive values of θ , or integrate θ by assuming successive values of v . That is, having the initial values at the origin, where $\theta = \phi$ and $v = V$, we can assume successively decreasing values of θ , say in $10'$ intervals ($\Delta\theta = 10'$), and determine the corresponding reductions in v (or Δv); by reducing v successively according to the Δv 's thus found, we can establish the value of v corresponding to any value of θ . Likewise, we can assume successively decreasing values of v (say $\Delta v = 10$ f.s.), and thus determine the value of θ corresponding to any value of v . Whichever of these procedures has been chosen, we can then apply a like process to each of the remaining differential equations. That is, if we have determined values of v corresponding to values of θ (from (505)), we can proceed to integrate x by applying to (502) successive small changes in θ and their corresponding changes in v . The variable chosen to be the one that is changed in successive regular intervals is called the *independent variable*.

Choice of independent variable

Solutions based on v as the independent variable are not convenient and are not used. The French have made extensive use of methods using θ as the independent variable, and their ballistic tables (including some which are used in this text) are based on this method. A number of variations also have been introduced, both here and abroad, in which the independent variable is not a simple element of the trajectory but a variable arbitrarily arrived at by certain transformations of the fundamental equations (not, however, by resorting to approximation devices such as Siacci's). The method introduced by Moulton is based on the time t as independent variable, and this method is used by the U. S. Army for the computation of its numerical integration ballistic tables. Moulton's Method (sometimes referred to also as the rectangular method) is by far the most obvious of the many methods based on numerical integration. It possesses the advantages of directness, simplicity, and accuracy to a high degree, although it does not lend itself to labor saving technique as well as do some of the others.* An elementary example of the application of numerical integration to the solution of a trajectory, using t as the independent variable, will be given presently. In the meantime we shall examine the general character of the process of numerical integration as applied to ballistics, by applying the process to some very simple cases.

604. Let us suppose that a projectile is fired practically horizontally, and that it is desired to determine how far the projectile will travel while its velocity is being reduced a given amount by air resistance. In order to confine our problem to simple terms, we shall assume a velocity reduction small enough to leave the path to be considered practically straight as well as horizontal—say a reduction from 1820 f.s. to 1780

Elementary solution with v as independent variable

* Examples of solutions using θ as independent variable, including a variation used by the French in the computation of their tables, are given in Chapters V and VI of *Ordnance Pamphlet No. 500* (U. S. Navy). For examples of solutions by Moulton's Method (using t as independent variable) and by Bennett's *tangent-reciprocal method* (based on an artificial variable) see *U. S. War Department Document No. 984* (1919).

f.s.—and also assume $C=1$. We shall now proceed to determine the distance covered by the projectile in steps each involving a reduction of 10 f.s., i.e., from 1820 to 1810, from 1810 to 1800, etc.

In the first of these steps the mean velocity of the projectile is 1815 f.s., and for this mean velocity the retardation according to the G -function is 435.35 f.s.s. (obtainable directly from Table I, since $C=1$). The time elapsing during the reduction of 10 f.s. in velocity from 1820 f.s. to 1810 f.s. very nearly equals the

reduction in velocity divided by the retardation, or $\frac{10 \text{ f.s.}}{435.35 \text{ f.s.s.}}$. The distance

covered during the same reduction very nearly equals the time elapsed multiplied by the mean velocity for the interval, or

$$\frac{10 \text{ f.s.}}{435.35 \text{ f.s.s.}} \times 1815 \text{ f.s.} = 41.7 \text{ feet.}$$

Thus we have determined that while its velocity is being reduced from 1820 f.s. to 1810 f.s., the projectile travels 41.7 feet.

For the second interval the reduction is from 1810 f.s. to 1800 f.s., the mean velocity is 1805 f.s., the retardation corresponding to the latter is 431.25 f.s.s., and the distance covered is

$$\frac{10}{431.25} \times 1805 = 41.9 \text{ feet.}$$

Proceeding similarly for the remaining intervals we obtain the results tabulated below.

Reduction in v (f.s.)	Distance covered (feet)
1820-1810	41.7
1810-1800	41.9
1800-1790	42.0
1790-1780	42.2
Total	167.8

605. As to the accuracy of the above determination, the following observations may be made. The use of the arithmetic mean of the retardation pertaining to the beginning and end of each 10 f.s. interval to find the time for that interval, and of a similarly determined mean velocity to find the distance covered, both are sources of inaccuracy, since the velocity-retardation relation is not a linear function. What degree of inaccuracy has been incurred by using 10 f.s. intervals can be determined very easily by making the computations for much smaller intervals. A determination of the distance covered for the velocity reduction from 1820 f.s. to 1810 f.s., in five 2 f.s. intervals, also gives 41.7 feet (in fact, the result obtained by using 2 f.s. intervals from 1820 f.s. to 1810 f.s. agrees to five places with that obtained in one step as above). Using but a single interval for the entire reduction from 1820 f.s. to 1780 f.s., the result obtained differs by less than a tenth of a foot from that obtained in four steps as above. (Taken to five places, the result obtained by using one 40 f.s. interval is 167.75 feet, compared to 167.78 feet as obtained by using four 10 f.s. intervals.)

It is apparent, then, that we can secure as great accuracy as may be desired, in this method, merely by regulating the size of the interval used in the inde-

pendent variable. In the specific instance defined by the above problem it appears that for the determination of distances, intervals of 10 f.s. afford accuracy to the hundredth of a foot, while for accuracy to the tenth of a foot intervals as great as 50 f.s. certainly might be used. In any case, the magnitude of the interval that may be used depends on the quantities involved and on the degree of accuracy desired, and it may be determined readily by a few trials. Inspection of the differences between the successive values obtained with any chosen interval gives a good indication of the degree of inaccuracy that is being incurred by the use of linear averages within the intervals. Inspection of the first differences in the tabulation given for the results in the above problem indicates at once that the inaccuracy due to the use of 10 f.s. intervals in this case is less than a tenth of a foot in each interval.

606. The above problem affords also a comparison, in very simple terms, between the process of determining elements of a trajectory by numerical integration and according to a tabulated retardation function (such as the G -function), and that of determining the same elements by formal integration and according to a retardation function of the character of Mayevski's. Siacci's Method embodies the

Comparison of
solution by
Siacci's Method
with solution
by numerical
integration

latter process. The solution, according to Siacci's Method, of the problem stated in the foregoing article, involves the solution of formula (519). Since we have assumed a trajectory that is practically horizontal and short enough to be considered a straight line, we have $\phi = \theta = 0$, whence $\beta = 1$, $f_0 = 1$, and also $u = v$. Then, since we also have assumed $C = 1$, the solution of (519) amounts merely to the integration of Siacci's space function (518) between the limits 1820 f.s. and 1780 f.s. These velocities do not all lie within any one of Mayevski's zones, and the integration of (518) must therefore be performed in two parts, the first from 1820 f.s. to 1800 f.s. and the second from 1800 f.s. to 1780 f.s., using the proper values of a and A in each case (art. 409). Substituting the required values of a and A , the process is expressed as follows (the numbers in parentheses are $\log A$).

$$x = - \frac{1}{(6.11926 - 10)} \int_{1800}^{1780} \frac{du}{u} + \frac{1}{(7.09620 - 10)} \int_{1820}^{1800} \frac{du}{u^{0.7}}$$

Performing these integrations, we have*

$$x = (-56872 + 56957) + (-25308 + 25392) = 169 \text{ feet.}$$

The result thus obtained does not differ greatly from that arrived at by the step-by-step process. The difference found is due partly to the fact that Mayevski's retardation functions, while representing good average relations for whole zones, cannot define the retardation for each velocity as accurately as is possible with a tabular function, and partly to the fact that the G -function and Mayevski functions are based on different experimental data and hence involve real differences, apart from such as may arise from the manner in which the functions are expressed. Although in the simple problem here considered the labor involved in the solution by Siacci's Method does not differ greatly from that involved in the numerical integration process, it is to be noted that the identical amount of labor, by Siacci's method, is sufficient to solve similar problems for any velocities within the same two zones (i.e., any from 2600 f.s. to 1370 f.s.), while by numerical inte-

* The integrals reduce, respectively, to $(4.24296) \log u$ and $(3.42668) u^{0.3}$, the numbers in parentheses being logarithms. According to Ingalls' Tables, based on computations correct to at least seven places, the result of this integration is 168.9 feet.

gration the labor, of course, increases greatly for more widely separated velocities.

607. The use of v as the independent variable offered no difficulties in the above problem, because definite limits of velocity could be assigned to each interval independently of any other variables entering into the solution, and hence the mean velocity and mean retardation for each interval could be found directly. In the more general case, v is modified also by a component of gravity depending upon the angle of inclination θ , while the latter, in turn, cannot be found without knowing v . In this case it is necessary to resort to successive approximations to establish the correct relations for each interval, and this applies whether the independent variable is v or θ , or any other element; but the use of v , in the general case, is less convenient than the use of θ or t . The character of the process that enters into the general case can be illustrated very simply as follows.

608. Let us suppose again that a projectile is fired practically horizontally, and that it is desired in this case to determine how far the projectile will travel during a given interval of time. In order to leave the problem in simple terms we shall again consider an interval short enough so that the portion of the trajectory considered will remain practically straight as well as horizontal,—and take $C=1$. Let us then determine the distance covered by the projectile in $\frac{1}{2}$ second, its initial velocity being 1800 f.s., and $C=1$, and proceed with the computations in five intervals of one-tenth of a second each.

Elementary
solution
with t as
independent
variable

We know that the initial velocity is 1800 f.s., and from Table I we find that at this velocity the retardation (for $C=1$) is 429.2 f.s.s. Then, as a first approximation, we can say that the reduction of velocity during the first tenth of a second is 42.92 f.s., and the remaining velocity at the end of this interval is $1800 - 42.9 = 1757.1$ f.s. A more accurate determination of the reduction of velocity during this interval may now be obtained by finding the retardation corresponding to the *mean* velocity in the interval, which is practically $\frac{1}{2}(1800 + 1757.1) = 1778.6$ f.s.

The corresponding retardation is 420.33 f.s.s., whence for one-tenth of a second the reduction in velocity is 42.03 f.s. A second approximation of the remaining velocity at the end of the interval is then $1800 - 42.0 = 1758.0$ f.s., and of the *mean* velocity for the interval $\frac{1}{2}(1800 + 1758.0) = 1779.0$ f.s. With the latter we now find a retardation of 420.49 f.s.s., whence the velocity reduction for one-tenth of a second is 42.05 f.s., and the remaining velocity at the end of the interval is $1800 - 42.0 = 1758.0$ f.s., which agrees with the next preceding approximation. Since there has been no change since the next preceding determination of the velocity for the end of the interval, there can be no further change in the value of the *mean* velocity last found, and we can take the latter to be 1779.0 f.s. We have now established the mean velocity for the first interval to the nearest tenth of a foot second, which is sufficiently accurate for our purpose. The distance covered during the interval is then found by multiplying this mean velocity by the time, thus $1779.0 \text{ f.s.} \times 0.1 \text{ sec.} = 177.9 \text{ feet.}$

Successive
approximations

Proceeding similarly with the second interval of one-tenth of a second, we start with the final approximation of the remaining velocity already found for the end of the first interval, or 1758.0 f.s. The corresponding retardation is 412.0 f.s.s., the reduction of velocity for one-tenth of a second is 41.20 f.s., and the first approximation of the remaining velocity at the end of the second interval is $1758.0 - 41.2 = 1716.8$ f.s., and of the mean velocity for this interval $\frac{1}{2}(1758.0 + 1716.8) = 1737.4$ f.s. A second approximation of the retardation is now found to be 403.6 f.s.s., of the reduction in velocity 40.36 f.s., of the remaining velocity $1758.0 - 40.4 = 1717.6$ f.s., and of the mean velocity $\frac{1}{2}(1758.0 + 1717.6) = 1737.8$ f.s. No further

change will be found in the latter, and the distance for the second interval is then $1737.8 \times 0.1 = 173.8$ feet. Carrying this process forward through the remaining three intervals, the results are as follows.*

<u>Time interval (seconds)</u>	<u>Distance covered (feet)</u>
0-0.1	177.9
0.1-0.2	173.8
0.2-0.3	169.8
0.3-0.4	166.0
0.4-0.5	162.4
Total	849.9

We also may tabulate the results more conveniently as follows, showing the remaining velocity, as well as the distance, at the end of each of the intervals.

<u>t (seconds)</u>	<u>v (f.s.)</u>	<u>x (feet)</u>
0	1800	0
0.1	1758.0	177.9
0.2	1717.6	351.7
0.3	1678.8	521.5
0.4	1641.5	687.5
0.5	1605.8	849.9

609. As to the accuracy of the above determination, it is found that by halving the size of the interval the results obtained do not differ from the above, while by doubling the interval the differences at the end of one-half second are about 0.1 f.s. in the velocity and about 0.2 feet in the distance. By computing the entire one-half second in a single interval, the differences noted are about 0.7 f.s. in the velocity and about 1.4 feet in the distance. Here again we find that by adjusting the size of the interval we can regulate the accuracy of the result according to our requirements.

The significant difference between the process used in the present case and that used in the case of the problem stated in article 604, is that in the present case it has been necessary to arrive at the mean retardation and mean velocity for each interval by a series of approximations, since the two are dependent upon each other.† In article 604, as already noted, this was unnecessary because the velocity limits of each interval were fixed definitely in advance.

610. We may now proceed to examine the essential features of the process of numerical integration as applied to the solution of a trajectory. The method that will be illustrated is based on the use of t as the independent variable and it is, fundamentally, the method that has been adopted by the U. S. Army for the computation of ballistic tables. For this method a very simple set of differential equations is set up as follows.

The retardation of a projectile due to air resistance, taking into account all known factors pertaining to the projectile itself and to the density of the air, is

* The process can, of course, be condensed somewhat. For example, the mean velocity for any interval of one-tenth of a second may be found more directly by subtracting one-twentieth of the retardation (in f.s.) from the initial velocity for that interval.

† This problem could have been solved more simply by using v as the independent variable, as in art. 604, for it will be recalled that the latter method yielded a value of t for the end of each interval. The more complicated process has been used here purposely, because it illustrates in simple terms the general nature of successive approximations within an interval.

expressed by the formula

$$E = \frac{G_v \times H_v}{C} \quad (408) \quad (601)$$

which has already been explained in detail in article 429. Now just as the remaining velocity at any point in the trajectory has the horizontal and vertical components, respectively, $v \cos \theta$ and $v \sin \theta$, the retardation due to air resistance has the components $E \cos \theta$ and $E \sin \theta$. The retardation of the projectile in the horizontal plane is due entirely to the air resistance; in the vertical plane, however, the projectile's initial velocity is retarded both by air resistance and by gravity. Since acceleration (or, as it is in this case, retardation) is the second derivative of distance with respect to time, the horizontal component of retardation of the

projectile is defined by $\frac{d^2x}{dt^2}$ and the vertical component by $\frac{d^2y}{dt^2}$. We may then

write for these retardations, including in the case of the vertical component the effect of gravity,

$$\begin{array}{l} \text{Differential} \\ \text{equations of} \\ \text{trajectory for} \\ \text{solution with} \\ \text{t as independent} \\ \text{variable} \end{array} \quad \frac{d^2x}{dt^2} = - E \cos \theta \quad (602)$$

$$\frac{d^2y}{dt^2} = - E \sin \theta - g. \quad (603)$$

Equations (601), (602), and (603) form the basis of the method of solution which is to be illustrated. In order to simplify the notation to be used in carrying out computations, we shall adopt the following.

- x = horizontal distance, or range, in feet.
- y = vertical distance, or ordinate, in feet.
- x' = horizontal velocity component, in f.s.
- y' = vertical velocity component, in f.s.
- x'' = horizontal retardation component, in f.s.s.
- y'' = vertical retardation component, in f.s.s.

In accordance with the above notation the entire system of formulas required in the solution may now be written as follows.

$$E = \frac{G_v \times H_v}{C} \quad (601)$$

$$x'' = - E \cos \theta \quad (602)$$

$$y'' = - E \sin \theta - g \quad (603)$$

$$x' = v \cos \theta \quad \text{or} \quad \cos \theta = \frac{x'}{v} \quad (604)$$

$$y' = v \sin \theta \quad \text{or} \quad \sin \theta = \frac{y'}{v} \quad (605)$$

$$v = \sqrt{(x')^2 + (y')^2}. \quad (606)$$

Also, in order to distinguish among values of the various elements at the various points through which the solution is to proceed, time subscripts will be used to

denote the point to which any value applies. For example, E_0 , x_0 , x_0' , x_0'' , y_0 , y_0' , y_0'' , v_0 , etc., will denote values at the point where $t=0$, i.e., at the origin; $E_{1/4}$, $x_{1/4}$, $x_{1/4}'$, $x_{1/4}''$, $y_{1/4}$, $y_{1/4}'$, $y_{1/4}''$, $v_{1/4}$, etc., will denote values at the point where $t=\frac{1}{4}$ second; and so on. The notations x_m' , x_m'' , y_m' , and y_m'' , will be used to denote the mean values of x' , x'' , y' , and y'' , respectively, for a given interval.

Elementary
example of
solution of
trajectory, with
 t as independent
variable

611. For illustration of the process of solution we will compute several points of the trajectory of the 16'' 2600 f.s. gun for the angle of departure 30° . The G - and H -tables are appended to this chapter for convenience in following this solution. The necessary initial data follow:

$$V = 2600 \text{ f.s.}, \quad \phi = 30^\circ, \quad d = 16'', \quad w = 2100 \text{ lbs.}, \quad i = .61, \quad \delta = 1.00$$

whence we have

$$C = \frac{w}{\delta i d^2} \quad (406)$$

$\delta = 1.00$	log 0.00000.....	colog 0.00000
$w = 2100$	log 3.32222	
$i = .61$	log 9.78533-10.....	colog 0.21467
$d^2 = 256$	log 2.40824.....	colog 7.59176-10
$C =$	log 1.12865	

and this value of C will not change during the computation.

From (604) and (605) we have $x' = v \cos \theta$ and $y' = v \sin \theta$, and since at the origin $\theta = \phi$ and $v = V$, we may compute the horizontal and vertical velocity components at the origin.

$v = V = 2600$	log 3.41497.....	log 3.41497
$\theta = \phi = 30^\circ$	lcos 9.93753-10.....	lsin 9.69897-10
$x_0' = 2251.6$	log 3.35250	
$y_0' = 1300.0$	log 3.11394	

Likewise we may compute the horizontal and vertical acceleration components at the origin, using (601), (602) and (603). At the origin $v = V = 2600$, hence we look up G_v in the G -table for the velocity 2600 f.s.; we find that it is 769.3 f.s.s. At the origin $y = 0$ and hence $H_v = 1$, or $\log H_v = 0.00000$, which we find also from the H -table for $y = 0$. Since $\theta = \phi = 30^\circ$, and C has already been computed, we have all the data required to solve (601) (602) and (603).

$G_v = 769.3$	log 2.88610	
H_v	log 0.00000	
C	log 1.12865.....	colog 8.87135-10
E_0	log 1.75745	
E_0	log 1.75745.....	log 1.75745
$\theta = \phi = 30^\circ$	lcos 9.93753-10.....	lsin 9.69897-10
$x_0'' = 49.54 \text{ f.s.s.}^*$	log 1.69498.....	
$E \sin \theta = 28.60 \text{ f.s.s.}$	log 1.45642	
$g = 32.16 \text{ f.s.s.}$		
$y_0'' = 60.76 \text{ f.s.s.}^*$		

* The minus signs are to be understood in applying these values.

FIRST INTERVAL

612 We now know that at the origin the horizontal velocity is $x' = 2251.6$ f.s. and that it is decreasing at this point at the rate of $x'' = 49.54$ f.s.s.; also that the vertical velocity is $y' = 1300$ f.s. and that it is decreasing at the rate of $y'' = 60.76$ f.s.s. If we assume a small interval of time for instance $\frac{1}{4}$ second, we can estimate what the velocity components will be at the end of that interval; they will be, approximately, the original values minus one fourth of the retardation for one second.

We can then make the predictions,

First Prediction

$$x_{1/4}' = x_0' - \frac{1}{4}x_0'' = 2251.6 - \frac{49.54}{4} = 2239.2 \text{ f.s.}$$

$$y_{1/4}' = y_0' - \frac{1}{4}y_0'' = 1300.0 - \frac{60.76}{4} = 1284.8 \text{ f.s.}$$

$$v_{1/4} = \sqrt{(x_{1/4}')^2 + (y_{1/4}')^2} = 2581.6 \text{ f.s.}$$

$$x_{1/4}' = 2239.2 \log 3.35009, 2\log 6.70018, (x_{1/4}')^2 = 5014000$$

$$y_{1/4}' = 1284.8 \log 3.10883, 2\log 6.21766, (y_{1/4}')^2 = 1650700$$

$$v_{1/4}^2 = \dots\dots\dots 6664700$$

$$6,664,700, \log 6.82378, \frac{1}{2}\log 3.41189, v_{1/4} = 2581.6$$

and from (604) and (605),

$$x_{1/4}' = 2239.2 \dots\dots\dots \log 3.35009$$

$$y_{1/4}' = 1284.8 \dots\dots\dots \log 3.10883$$

$$v_{1/4} = 2581.6 \dots\dots \log 3.41189, \text{colog } 6.58811 - 10 \dots\dots \text{colog } 6.58811 - 10$$

$$\theta_{1/4} = 29^\circ 50' 46'' \dots\dots\dots \text{colog } 9.93820 - 10 \dots\dots \text{lsin } 9.69694 - 10$$

These predictions would be accurate only if the retardation had remained constant throughout the $\frac{1}{4}$ -second interval, which we know is not the case. A more nearly correct value of the retardation may be obtained by taking the mean of the values for the beginning and end of the interval. We have already found what the retardation components are for the beginning of the interval; for the end of the interval we must solve for $x_{1/4}''$ and $y_{1/4}''$, and to do this we will have to use the approximate data we have already found for the end of the interval.

Second Prediction

613. The approximate mean vertical velocity for the interval is evidently the mean of y_0' and $y_{1/4}'$, whence,

$$y_m' = \frac{1}{2}(y_0' + y_{1/4}') = \frac{1}{2}(1300.0 + 1284.8) = \frac{1}{2} \times 2584.8 = 1292.4 \text{ f.s.}$$

and the approximate height of the projectile at the end of the interval is given by the product of the mean vertical velocity and the time interval, whence,

$$y_{1/4} = \frac{1}{2} \times 1292.4 = 323.1 \text{ feet}$$

Entering the H -table with 323 feet we find $\log H_y = 9.99557 - 10$. And with the velocity we found existing at the end of the interval, 2581.6 f.s., we enter the G -table and find $G_y = 760.7$. We may now solve (601), (602) and (603) for $x_{1/4}''$ and $y_{1/4}''$.

$$G_y = 760.7 \dots\dots\dots \log 2.88121$$

$$H_y \dots\dots\dots \log 9.99557 - 10$$

$$C \dots\dots\dots \text{colog } 8.87135 - 10$$

$$E_{1/4} \dots\dots\dots \log 1.74813 \dots\dots\dots \log 1.74813$$

$$\theta_{1/4} = 29^\circ 50' 46'' \dots\dots\dots \text{colog } 9.93821 - 10 \dots\dots \text{lsin } 9.69694 - 10$$

$$x_{1/4}'' = 48.57 \text{ f.s.s.} \dots\dots\dots \log 1.68634 \dots\dots\dots$$

$$27.87 \dots\dots\dots \log 1.44507$$

$$32.16$$

$$y_{1/4}'' = 60.03 \text{ f.s.s.}$$

We now have the retardation components for the beginning of the interval (x_0'' and y_0'') and also the approximate values for the end of the interval ($x_{1/4}''$ and $y_{1/4}''$), and we may proceed to correct our first estimate of the velocity components at the end of the interval.

The mean retardation components for the interval are evidently,

$$x_m'' = \frac{1}{2}(x_0'' + x_{1/4}'') = \frac{1}{2}(49.54 + 48.57) = 49.06 \text{ f.s.}$$

$$y_m'' = \frac{1}{2}(y_0'' + y_{1/4}'') = \frac{1}{2}(60.76 + 60.03) = 60.40 \text{ f.s.}$$

and we may then predict that the velocity components at the end of the $\frac{1}{4}$ -second interval will be

$$x_{1/4}' = x_0' - \frac{1}{2}x_m'' = 2251.6 - \frac{1}{2} \times 49.06 = 2239.3 \text{ f.s.}$$

$$y_{1/4}' = y_0' - \frac{1}{2}y_m'' = 1300.0 - \frac{1}{2} \times 60.40 = 1284.8 \text{ f.s.}$$

For v we have, as before,

$$v_{1/4} = \sqrt{(x_{1/4}')^2 + (y_{1/4}')^2} = 2581.8 \text{ f.s.}$$

$$x_{1/4}' = 2239.3 \dots \log 3.35011 \dots 2 \log 6.70022, (x_{1/4}')^2 = 5,014,400$$

$$y_{1/4}' = 1284.9 \dots \log 3.10887 \dots 2 \log 6.21774, (y_{1/4}')^2 = 1,651,000$$

$$(v_{1/4}')^2 \dots \dots \dots 6,665,400$$

$$6,665,400, \log 6.82383, \frac{1}{2} \log 3.41192, \dots \dots \dots v_{1/4} = 2581.8 \text{ f.s.}$$

For θ we have, using (604) and (605) as before,

$$x_{1/4}' = 2239.3 \dots \log 3.35011$$

$$y_{1/4}' = 1284.9 \dots \dots \dots \log 3.10887$$

$$v_{1/4}' = 2581.8 \dots \log 3.41192 \dots \dots \dots \log 6.58808 - 10 \dots \dots \dots \log 6.58808 - 10$$

$$\theta_{1/4} = 29^\circ 50' 44'' \dots \dots \dots \cos 9.93819 - 10 \dots \dots \dots \sin 9.69695 - 10$$

Comparing the results of the second prediction with those of the first we find very close agreement in all of the elements found. The closeness of the first prediction may be attributed to the very small time interval used. Had the time interval been larger we would probably have been obliged to make several predictions before coming close to the correct result.

Limit of accuracy It is customary to carry on the process until two successive sets of values of the retardation components agree to the hundredth of a foot-second-second. With five-place tables we could carry the process on for accuracy to the thousandth of a foot-second-second, but the degree of accuracy obtained by two decimal places in the retardation components is quite sufficient for any purpose. For velocity components, and for x and y , one decimal is sufficient.

We will make one more prediction in order to be certain that there will be no further change in the computed elements.

Third Prediction

614. The corrected estimate of the mean vertical velocity becomes,

$$v_m' = \frac{1}{2}(1300.0 + 1284.9) = 1292.4 \text{ f.s.}$$

and hence the height of the projectile at the end of the interval becomes,

$$y_{1/4} = \frac{1}{2} \times 1292.4 = 323.1 \text{ feet}$$

From the H -table we find, with 323 feet, that $\log H_y = 9.99557 - 10$. And with the previously found velocity of 2581.8 f.s. we find in the G -table that $G_y = 760.8$.

The data for solving (601), (602), and (603), are now complete, and we proceed as before,

$$G_y = 760.8 \dots \log 2.88127$$

$$H_y \dots \log 9.99557 - 10$$

$$C \dots \dots \dots \log 8.87135 - 10$$

$$E_{1/4} \dots \log 1.74819 \dots \log 1.74819$$

$$\theta_{1/4} \dots \dots \dots \cos 9.93821 - 10 \dots \dots \dots \sin 9.69693 - 10$$

$$x_{1/4}'' = 48.57 \text{ f.s.} \dots \log 1.68640$$

$$27.57 \dots \dots \dots \log 1.44512$$

$$32.16$$

$$y_{1/4}'' = 60.03 \text{ f.s.}$$

615. We now find exact agreement between the last two successive predictions of x'' and y'' . We may be certain, then, that any further continuation of the predictions will be merely a repetition of previous work, and that no further accuracy within the chosen limits may be obtained.

We complete the operation for the first interval by finding x . The horizontal velocity at the origin (x_0) was 2251.6 f.s.; at the end of $\frac{1}{4}$ second it was $x_{1/4}' = 2239.3$ f.s., this being the value from the last prediction. Hence the mean horizontal velocity for the interval was,

$$x_m' = \frac{1}{2}(2251.6 + 2239.3) = 2245.4 \text{ f.s.}$$

and the horizontal distance traveled by the projectile in $\frac{1}{4}$ second was,

$$x_{1/4} = \frac{1}{4} \times 2245.4 = 561.4 \text{ feet}$$

SECOND INTERVAL

616. The entire process of locating the position of the projectile at the end of an interval of time of $\frac{1}{4}$ second has been illustrated in the preceding articles. Before proceeding further it will be of advantage to arrange the information already obtained in a systematic manner, as follows:

$$V = 2600 \text{ f.s.}$$

$$\phi = 30^\circ$$

$$\log C = 1.12865$$

t	v	θ	x	x'	x''	$\Delta x''$	y	y'	y''	$\Delta y''$
0	2600 0	30° 00' 0	0	2251.6	49.54	0	1300 0	60.76	...
$\frac{1}{4}$	2581.8	29° 50' 7	561.4	2239.3	48 57	0.97	323.1	1284.9	60.03	0.73
$\frac{1}{2}$	(2564 0)	(29° 41' 5)	.	(2227.3)	(47 60)	(0 97)	(642.5)	(1270.0)	(59.30)	(0.73)

First Prediction

We may now proceed to a second $\frac{1}{4}$ -second interval, starting from the point already found, and we will then have the location of the projectile at the end of $\frac{1}{2}$ second. But our first prediction for the new interval may be made on the basis of information gained in the first interval. For instance, from our tabulation above we find that x'' decreased 0.97 f.s.s. in the first $\frac{1}{4}$ second, and that y'' decreased 0.73 f.s.s.; these quantities are the first differences in x'' and y'' columns, and they have been denoted by $\Delta x''$ and $\Delta y''$. Assuming these changes to remain constant for the next interval we may predict that at the end of another $\frac{1}{4}$ second we will have $x_{1/2}'' = (48.57 - 0.97) = 47.60$, and $y_{1/2}'' = (60.03 - 0.73) = 59.30$. These predicted values have been entered in the above tabulation in brackets.

The approximate mean values of x'' and y'' for the interval $\frac{1}{4} - \frac{1}{2}$ second will evidently be,

$$x_m'' = \frac{1}{2}(48.57 + 47.60) = 48.08 \text{ f.s.s.}$$

$$y_m'' = \frac{1}{2}(60.03 + 59.30) = 59.66 \text{ f.s.s.}$$

and the approximate values of x' and y' at the end of the interval will be,

$$x_{1/2}' = 2239.3 - \frac{1}{4} \times 48.08 = 2227.3 \text{ f.s.}$$

$$y_{1/2}' = 1284.9 - \frac{1}{4} \times 59.66 = 1270.0 \text{ f.s.}$$

Likewise the approximate mean value of the vertical velocity for the new interval will be,

$$y_m' = \frac{1}{2}(1284.9 + 1270.0) = 1277.4 \text{ f.s.}$$

and the vertical distance traveled by the projectile during the $\frac{1}{4}$ second will be,

$$\Delta y = \frac{1}{4} \times 1277.4 = 319.4 \text{ feet}$$

and hence its height at the end of the new interval will be,

$$y_{1/2} = y_{1/4} + 319.4 = 323.1 + 319.4 = 642.5 \text{ feet}$$

For v and θ we use (604), (605) and (606) as usual. For convenience we will drop the time subscripts except for the quantity we are finding.

$$x' = 2227.3, \log 3.34778, 2\log 6.69556, (x')^2 = 4,960,900$$

$$y' = 1270.0, \log 3.10380, 2\log 6.20760, (y')^2 = 1,612,900$$

$$v_{1/2}^2 \dots \dots \dots 6,573,800$$

$$6,573,800, \log 6.81782, \frac{1}{2}\log 3.40891, v_{1/2} = 2564.0 \text{ f.s.}$$

$$\begin{aligned}
 x' &= 2227.3 \dots \log 3.34778 \\
 y' &= 1270.0 \dots \log 3.10380 \\
 v &= 2564.0 \dots \log 3.40891 \dots \text{colog } 6.59109 - 10 \dots \text{colog } 6.59109 - 10 \\
 \theta_{1/2} &= 29^\circ 41' 27'' \dots \text{cos } 9.93887 - 10 \dots \text{sin } 9.69489 - 10
 \end{aligned}$$

The approximate values we have found constitute the *first predictions* for the $\frac{1}{2}$ -second point, and they have been entered in the above tabulation in brackets. We may proceed to correct these approximate values just as we did in the first interval.

Second Prediction

617. With $y = 642$ we find from the H -table that $\log H_y = 9.99119 - 10$; and with $v = 2564.0$ we find from the G -table that $G_v = 752.4$, whence we find from (601), (602) and (603),

$$\begin{aligned}
 G_v &= 752.4 \dots \log 2.87845 \\
 H_y &\dots \log 9.99119 - 10 \\
 C &\dots \text{colog } 8.87135 - 10 \\
 E_{1/2} &\dots \log 1.73899 \dots \log 1.73899 \\
 \theta_{1/2} &= 29^\circ 41' 27'' \dots \text{cos } 9.93888 - 10 \dots \text{sin } 9.69489 - 10 \\
 x_{1/2}'' &= 47.63 \text{ f.s.} \dots \log 1.67787 \\
 27.16 &\dots \log 1.43388 \\
 32.16 & \\
 y_{1/2}'' &= 59.32 \text{ f.s.}
 \end{aligned}$$

We will now correct all of the remaining values, and since the process is exactly the same as in the preceding article no further explanations are required.

$$\begin{aligned}
 x_m'' &= \frac{1}{2}(48.57 + 47.63) = 48.10 \text{ f.s.} \\
 y_m'' &= \frac{1}{2}(60.03 + 59.32) = 59.68 \text{ f.s.} \\
 x_{1/2}' &= 2239.3 - \frac{1}{2} \times 48.10 = 2227.3 \text{ f.s.} \\
 y_{1/2}' &= 1284.9 - \frac{1}{2} \times 59.68 = 1270.0 \text{ f.s.} \\
 y_m' &= \frac{1}{2}(1284.9 + 1270.0) = 1277.4 \text{ f.s.} \\
 \Delta y &= \frac{1}{2} \times 1277.4 = 319.4 \text{ ft.} \\
 y_{1/2} &= 323.1 + 319.4 = 642.5 \text{ ft.}
 \end{aligned}$$

Since the values of $x_{1/2}'$ and $y_{1/2}'$ have not changed from the previous prediction, $v_{1/2}$ and $\theta_{1/2}$ also will not change, and hence, from the first prediction,

$$v_{1/2} = 2564.0 \text{ f.s.}, \quad \theta_{1/2} = 29^\circ 41' 27''$$

It will be observed that the values obtained in the first and second predictions agree within the required limits of accuracy for the elements x' , y' , y , v and θ , and hence we have reached the limit of accuracy for these elements. The second predictions of x'' and y'' vary considerably from the first predictions and it would appear that we should make a further prediction for these elements. However, none of the quantities entering into the computation of x'' and y'' (by (601), (602) and (603)) have changed since the previous prediction, and hence the results of any further prediction could not possibly be different from the results of the previous prediction. It follows that we have also reached the limit of accuracy for x'' and y'' .

We may now complete the work of the second interval by computing the value of x for the $\frac{1}{2}$ -second point. Proceeding as in article 615 (par. 2),

$$x_m' = \frac{1}{2}(2239.3 + 2227.3) = 2233.3 \text{ f.s.}$$

whence the increase in x during the interval $\frac{1}{2} - \frac{1}{2}$ second was,

$$\Delta x = \frac{1}{2} \times 2233.3 = 558.3 \text{ feet}$$

and the value of x at the end of $\frac{1}{2}$ second becomes,

$$x_{1/2} = 561.4 + 558.3 = 1119.7 \text{ feet}$$

It will be noted that there would be no purpose in making preliminary predictions for x , since x does not enter into the other quantities. We must make successive predictions of y , θ , v , etc., because the other elements depend upon these. In the case of x we may make our final and only computation after the final values of the other elements have been found.

THIRD INTERVAL

616. We now tabulate our data once more, as follows:

t	s	θ	s	x'	x''	Δ_1	Δ_2	y	y'	y''	Δ_1	Δ_2
0	2600.0	30° 00'.0	0	2251.6	49.54	0	1200.0	60.76
$\frac{1}{4}$	2581.0	29 50'.7	561.4	2229.3	48.57	0.97	323.1	1284.9	60.03	0.73
$\frac{1}{2}$	2564.0	29 41'.4	1119.7	2227.3	47.63	0.94	0.03	642.5	1270.0	59.32	0.71	0.03
$\frac{3}{4}$	(2546.4)	(29 32'.2)	..	(2215.5)	(46.72)	(0.91)	(0.03)	(968.1)	(1255.3)	(58.63)	(0.69)	(0.03)

First Prediction

Having three quantities in the x'' and y'' columns, we are able to find the *first and second differences for x'' and y''* , i.e., *the change and rate of change in x'' and y'' for $\frac{1}{4}$ -second intervals*. The first and second differences have been denoted by Δ_1 and Δ_2 , respectively. Assuming now that the *rates of change* (i.e., Δ_2) remain the same for the next $\frac{1}{4}$ second, we predict the *changes* (i.e., Δ_1) during the next $\frac{1}{4}$ second will be $(0.94-0.03)=0.91$ for x'' , and $(0.71-0.02)=0.69$ for y'' , and these predicted values have been entered **Making use of second differences** in the above tabulation in brackets. We proceed to predict the values of x'' and y'' from these changes to be $x_{\frac{1}{4}}''=(47.63-0.91)=46.72$, and $y_{\frac{1}{4}}''=(59.32-0.69)=58.63$, and these predicted values have also been entered in brackets.*

The first prediction is completed just as in article 616.

$$\begin{aligned}
 x_m'' &= \frac{1}{4}(47.63+46.72) = 47.18 \text{ f.s.s.} \\
 y_m'' &= \frac{1}{4}(59.32+58.63) = 58.98 \text{ f.s.s.} \\
 x_{\frac{1}{4}}' &= 2227.3 - \frac{1}{4} \times 47.18 = 2215.5 \text{ f.s.} \\
 y_{\frac{1}{4}}' &= 1270.0 - \frac{1}{4} \times 58.98 = 1255.3 \text{ f.s.} \\
 y_m' &= \frac{1}{4}(1270.0+1255.3) = 1262.6 \text{ f.s.} \\
 \Delta y &= \frac{1}{4} \times 1262.6 = 315.6 \text{ ft.} \\
 y_{\frac{3}{4}} &= 642.5+315.6 = 958.1 \text{ ft.}
 \end{aligned}$$

$$\begin{aligned}
 x' &= 2215.5, \log 3.34547, 2\log 6.69094, (x')^2 = 4,908,400 \\
 y' &= 1255.3, \log 3.09875, 2\log 6.19750, (y')^2 = 1,575,800
 \end{aligned}$$

$$\begin{aligned}
 v_{\frac{1}{4}}^2 & \dots \dots \dots 6,484,200 \\
 6,484,200, \log 6.81185, \frac{1}{2}\log 3.40592, v_{\frac{1}{4}} &= 2546.4 \text{ f.s.}
 \end{aligned}$$

$$\begin{aligned}
 x' &= 2215.5 \dots \dots \dots \log 3.34547 \\
 y' &= 1255.3 \dots \dots \dots \log 3.09875 \\
 v &= 2546.4, \log 3.40592 \dots \dots \log 6.59408-10 \dots \dots \log 6.59408-10 \\
 \theta_{\frac{1}{4}} &= 29^\circ 32' 11'' \dots \dots \dots \log 9.93955-10 \dots \dots \log 9.69283-10
 \end{aligned}$$

Second Prediction

Proceeding as in article 617, we find from the H -table, with $y=958$ feet, that $\log H_y=9.98686-10$, and from the G -table, with $v=2546.4$ f.s., that $G_v=744.4$.

$$\begin{aligned}
 G_v &= 744.4 \dots \dots \dots \log 2.87181 \\
 H_y & \dots \dots \dots \log 9.98686-10 \\
 C & \dots \dots \dots \log 8.87135-10 \\
 E_{\frac{1}{4}} & \dots \dots \dots \log 1.73002 \dots \dots \log 1.73002 \\
 \theta_{\frac{1}{4}} &= 29^\circ 32' 11'' \dots \dots \dots \log 9.93954-10 \dots \dots \log 9.69283-10 \\
 x_{\frac{1}{4}}'' &= 46.73 \text{ f.s.s.} \dots \dots \dots \log 1.66956 \\
 26.48 & \dots \dots \dots \log 1.42285 \\
 32.16 & \\
 y_{\frac{1}{4}}'' &= 58.64 \text{ f.s.s.}
 \end{aligned}$$

* See article 622.

The value of y_1'' from the second prediction agrees exactly with that from the first; in the case of x_1'' a difference of .01 f.s.s. will be noted. It is apparent that a difference of .01 f.s.s. in x'' will cause a difference of only .0025 f.s. in x' during the $\frac{1}{2}$ -second interval, which is beyond the limits we have chosen. Hence our first predictions were correct for all elements except x'' and for the latter the second prediction is correct.

To find x_1 we have,

$$\begin{aligned}x_m' &= \frac{1}{2}(2227.3 + 2215.5) = 2221.4 \text{ f.s.} \\ \Delta x &= \frac{1}{2} \times 2221.4 = 555.4 \text{ feet} \\ x_1 &= 1119.7 + 555.4 = 1675.1 \text{ feet}\end{aligned}$$

FOURTH INTERVAL

619. For the next $\frac{1}{2}$ -second interval we will proceed exactly as in the previous interval, and will carry out the process without interposing any further explanation.

t	r	θ	x	x'	x''	Δ_1	Δ_2	y	y'	y''	Δ_1	Δ_2
0	2800.0	30°00'.0	0	2251.6	49.54	0	1300.0	60.76
$\frac{1}{4}$	2581.9	29 50'.7	561.4	2239.3	48.57	0.97	223.1	1284.9	60.03	0.73
$\frac{1}{2}$	2564.0	29 41'.4	1119.7	2227.3	47.63	0.94	0.03	642.5	1270.0	59.32	0.71	0.02
$\frac{3}{4}$	2546.4	29 32'.2	1675.1	2215.5	46.73	0.90	0.04	958.0	1255.3	58.64	0.68	0.03
1	(2529.2)	(29 22'.6)	2227.5	(2203.9)	(45.87)	(0.86)	(0.04)	(1270.0)	(1240.7)	(57.99)	(0.65)	(0.03)

First Prediction

$$\begin{aligned}x_1'' &= 46.73 - 0.86 = 45.87 \\ y_1'' &= 58.64 - 0.65 = 57.99 \\ x_m'' &= \frac{1}{2}(46.73 + 45.87) = 46.30 \text{ f.s.s.} \\ y_m'' &= \frac{1}{2}(58.64 + 57.99) = 58.32 \text{ f.s.s.} \\ x_1' &= 2215.5 - \frac{1}{2} \times 46.30 = 2203.9 \text{ f.s.} \\ y_1' &= 1255.3 - \frac{1}{2} \times 58.30 = 1240.7 \text{ f.s.} \\ y_m' &= \frac{1}{2}(1255.3 + 1240.7) = 1248.0 \text{ f.s.} \\ \Delta y &= \frac{1}{2} \times 1248.0 = 312.0 \text{ ft.} \\ y_1 &= 958.0 + 312.0 = 1270.0 \text{ ft.}\end{aligned}$$

$$\begin{aligned}x' &= 2203.9, \log 3.34319, 2\log 6.68638, (x')^2 = 4,857,100 \\ y' &= 1240.7, \log 3.09367, 2\log 6.18734, (y')^2 = 1,539,400 \\ v_1^2 &= 6,396,500 \\ 6,396,500, \log 6.80595, \frac{1}{2}\log 3.40298, v_1 &= 2529.2 \text{ f.s.}\end{aligned}$$

$$\begin{aligned}x' &= 2203.9 \dots \log 3.34319 \\ y' &= 1240.7 \dots \log 3.09367 \\ v_1 &= 2529.2 \dots \log 3.40298 \dots \colog 6.59702 - 10 \dots \colog 6.59702 - 10 \\ \theta_1 &= 29^\circ 22' 38'' \dots \lrcorner \cos 9.94021 - 10 \dots \lrcorner \sin 9.69069 - 10\end{aligned}$$

Second Prediction

From H -table, with 1270 feet, $\log H_y = 9.98258 - 10$

From G -table, with 2529.2 f.s., $G_s = 736.4$

$$\begin{aligned}G_s &= 736.4 \dots \log 2.86711 \\ H_y &\dots \log 9.98258 - 10 \\ C &\dots \colog 8.87135 - 10\end{aligned}$$

$$\begin{aligned}\bar{M}_1 &\dots \log 1.72104 \dots \log 1.72104 \\ \theta_1 &= 29^\circ 22' 38'' \dots \lrcorner \cos 9.94023 - 10 \dots \lrcorner \sin 9.69069 - 10\end{aligned}$$

$$\begin{aligned}x_1'' &= 45.84 \text{ f.s.s.} \dots \log 1.66127 \\ &\hline &25.81 \dots \log 1.41173 \\ &32.16 \\ v_1'' &= 57.97 \text{ f.s.s.}\end{aligned}$$

The difference between the first and second predictions is .03 f.s. in x'' ; the change caused in x' by this amount would be less than .01 f.s., hence the first predictions were correct for all elements except x'' , and for the latter the second prediction is correct. Finally, we have,

$$\begin{aligned}x_m' &= \frac{1}{2}(2215.5 + 2203.9) = 2209.7 \text{ f.s.} \\ \Delta x &= \frac{1}{2} \times 2209.7 = 552.4 \text{ feet} \\ x_1 &= 1675.1 + 552.4 = 2227.5 \text{ feet}\end{aligned}$$

620. We have now established the position of the projectile at four points separated by equal increments of time. With the information we have gained of the rate of change of the various elements, it is possible to increase the interval, and it will be most convenient to *double* it. In order to use the information already obtained we must combine it into two equal intervals, which means in this case two $\frac{1}{2}$ -second intervals. This is done simply by tabulating the data for the points at times 0, $\frac{1}{2}$ and 1, as already found. The tabulation follows:

FIFTH INTERVAL

The work for the $\frac{1}{2}$ -second interval is identical with that for the $\frac{1}{2}$ -second intervals except that we must, of course, compute changes for $\frac{1}{2}$ second instead of $\frac{1}{4}$ second.

t	v	θ	x	x'	x''	Δ_1	Δ_2	y	y'	y''	Δ_1	Δ_2
0	2600.0	30° 00'.0	0	2251.6	49.54	0	1300.0	80.76
$\frac{1}{2}$	2584.0	29 41'.4	1119.7	2227.3	47.63	1.91	...	642.5	1270.0	59.32	1.44	...
1	2529.2	29 22'.6	2227.5	2203.9	45.84	1.79	0.12	1270.0	1240.7	57.97	1.35	0.09
$1\frac{1}{2}$	(2495.5)	(29 03' 7)	3323.8	(2181.4)	(44.17)	(1.67)	(0.12)	(1893.2)	(1212.0)	(56.71)	(1.26)	(0.09)

First Prediction

$$\begin{aligned}x_{3/2}'' &= 45.84 - 1.67 = 44.17 \\ y_{3/2}'' &= 57.97 - 1.26 = 56.71 \\ x_m'' &= \frac{1}{2}(45.84 + 44.17) = 45.00 \text{ f.s.} \\ y_m'' &= \frac{1}{2}(57.97 + 56.71) = 57.34 \text{ f.s.} \\ x_{3/2}' &= 2203.9 - \frac{1}{2} \times 45.00 = 2181.4 \text{ f.s.} \\ y_{3/2}' &= 1240.7 - \frac{1}{2} \times 57.34 = 1212.0 \text{ f.s.} \\ y_m' &= \frac{1}{2}(1240.7 + 1212.0) = 1226.4 \text{ f.s.} \\ \Delta y &= \frac{1}{2} \times 1226.4 = 613.2 \text{ ft.} \\ y_{3/2} &= 1270.0 + 613.2 = 1883.2 \text{ ft.}\end{aligned}$$

$$\begin{aligned}x' &= 2181.4, \log 3.33874, 2\log 6.67748, (x')^2 = 4,758,600 \\ y' &= 1212.0, \log 3.08350, 2\log 3.16700, (y')^2 = 1,468,900 \\ v_{3/2}^2 &= \dots\dots\dots 6,227,500 \\ 6,227,500, \log 6.79432, \frac{1}{2}\log 3.39716, v_{3/2} &= 2495.5 \text{ f.s.}\end{aligned}$$

$$\begin{aligned}x' &= 2181.4 \dots\dots\dots \log 3.33874 \\ y' &= 1212.0 \dots\dots\dots \dots\dots \log 3.08350 \\ v_{3/2} &= 2495.5 \dots\dots \log 3.39716 \dots\dots \log 6.60284 - 10 \dots\dots \log 6.60284 - 10 \\ \theta_{3/2} &= 29^\circ 03' 23'' \dots\dots\dots \cos 9.94158 - 10 \dots\dots \sin 9.68634 - 10\end{aligned}$$

Second Prediction

From H -table, with 1883 feet, $\log H_s = 9.97417 - 10$

From G -table, with 2495.5 f.s., $G_s = 721.2$

$$\begin{aligned}G_s &= 721.2 \dots\dots\dots \log 2.85806 \\ H_s &\dots\dots\dots \log 9.97417 - 10 \\ C &\dots\dots\dots \log 8.87135 - 10 \\ E_{3/2} &\dots\dots\dots \log 1.70358 \\ \theta_{3/2} &= 29^\circ 03' 23'' \dots\dots\dots \cos 9.94158 - 10 \dots\dots \sin 9.68634 - 10 \\ x_{3/2}'' &= 44.17 \text{ f.s.} \dots\dots\dots \log 1.64516 \\ 24.54 &\dots\dots\dots \log 1.38992 \\ 32.16 &\dots\dots\dots \\ v_{3/2}'' &= 56.70 \text{ f.s.}\end{aligned}$$

Since no further changes will be found; due to the close agreement between the two predictions of x'' and y'' , we complete the interval by finding x .

$$\begin{aligned} x_m' &= \frac{1}{2}(2203.9 + 2181.4) = 2192.6 \text{ f.s.} \\ \Delta x &= \frac{1}{2} \times 2192.6 = 1096.3 \text{ feet} \\ x_{2.2} &= 2227.5 + 1096.3 = 3323.8 \text{ feet} \end{aligned}$$

621. By continuing the process outlined above the entire trajectory may be solved. The following additional features are encountered as the solution progresses.

Finding the summit (a) The summit of the trajectory is defined by the point where $y' = 0$, $x' = v$, and $\theta = 0$; the value of y at this point is the maximum ordinate y_s . It will usually be necessary to interpolate within one of the chosen intervals in order to determine exactly where this occurs. Beyond the summit, y' , Δy , and θ all become negative, and y' increases, y decreases, and θ increases. With x' continuing to decrease, while y' increases, a point is reached in the descending branch where the increases in y' just balance the decreases in x' (i.e., in formula (606)); at this point v has its least value and beyond it v increases.

Finding the point of fall (b) The point of fall is defined by the point where y again becomes zero. Interpolation within the last interval will probably be necessary to determine exactly where this occurs. The values of t , v , θ , and x corresponding to $y = 0$ are, respectively, T , v_ω , ω , and X for the trajectory.

622. The portion of the illustrative problem actually solved above demonstrates adequately the principles involved in the solution of an entire trajectory. In order not to obscure the essential features of the method by details of labor-saving technique, the process illustrated has been left in a very elementary stage

Devices for shortening the computational work insofar as computational features are concerned. Considering that trajectories involving times of flight up to about 100 seconds occur in practice, it is obvious that the shortening of the process of solution is an important consideration. The principal features

leading to a shortening of the process are as follows:

(a) The integrations are carried forward by the use of difference formulas,* which lead more directly to the successive points.

(b) The size of the interval is increased as the work progresses. This is generally accomplished by doubling the size of the interval successively as soon as second differences for the increased size of interval become available (as illustrated in article 620). This successive doubling of the interval is generally carried on until the interval is 2 seconds; sometimes 4-second intervals are used, although the latter are rather large for trajectory computations. The maximum size of the interval is generally governed by the consideration that the required degree of accuracy in the predictions be obtained by the use of no higher than second differences.

* The use of difference formulas is covered in most treatises on the solution of differential equations. A concise exposition is given on pp. 115 and 159 of Marks' *Mechanical Engineers' Handbook*, Second Edition. Further demonstrations of the solution of trajectories by numerical integration are given in the following references: Chapter III, *New Methods in Exterior Ballistics*, F. R. Moulton; *The Method of Numerical Integration in Exterior Ballistics*, Dunham Jackson (U. S. War Department Document No. 984, 1919); *A Course in Exterior Ballistics*, R. S. Hoar (U. S. War Department Document No. 1051, 1920).

623. Since the gravitational constant g varies according to latitude and height above the earth's surface, it is clear that both of these considerations enter into the solution of a trajectory by affecting the value of g used therein. We will examine briefly the complications that are introduced into our problem by these variations in g , and what degree of error results from neglecting them.

It is true, strictly speaking, that a trajectory solution which is correct for one locality cannot also be correct for another having a different latitude. **Effect of change in g with latitude** The sea-level value of g varies from 32.087 f.s.s. at the equator to 32.257 f.s.s. at the poles.* The value $g = 32.16$ f.s.s. (corresponding to sea level at latitude about 41°) is used by the U. S. Navy for trajectory computations involving g directly; at the U. S. Naval Proving Ground, $g = 32.155$ f.s.s. is used in experimental work. The U. S. Army uses $g = 9.80$ m.s.s. $= 32.152$ f.s.s.; the French generally use $g = 9.81$ m.s.s. $= 32.185$ f.s.s., but for close work sometimes use $g = 9.8085$ m.s.s. $= 32.180$ f.s.s. (all of these are values at sea level). But when a value of i is determined from experimental firing, as will be outlined in the next article, the value of i thus obtained necessarily accounts for any difference between the values of g as assumed in the computation of the ballistic tables used in this determination, and as actually existing at the locality of the experimental firing. The range table computed according to this value of i and with the same ballistic tables is thus adjusted, in effect, for the value of g actually existing at the locality at which the experimental determination of i was carried out, regardless of what may have been the value of g assumed for the ballistic tables. A correction might now be deduced for adjusting the range-table values of elements of the trajectory for the change in g with latitude, but this is not considered necessary, in view of the small changes involved. Such changes in g as occur within the extreme limits of latitude involved in practice are of small importance, especially in comparison with such errors as may exist in the retardation function itself.

The situation with respect to the variation of g with altitude is much the same. The differences between values of g at the surface and at the greatest ordinates occurring in artillery practice remain small enough to be inconsiderable in comparison with the probable inaccuracies of the retardation function. A further inaccuracy results from considering g to act always parallel to the original Y -axis, whereas g actually acts always directly toward the center of the earth, i.e., vertically with respect to the curved surface of the earth rather than with respect to the flat plane tangent to the earth at the origin of the trajectory. But the change in direction of g with respect to the assumed flat plane is so small (being only about 1° in 120,000 yards) that the error resulting from neglecting it is very small. Moreover, both of these slight inaccuracies also are eventually accounted for in the value of i .

624. Having arrived at the conclusion, in article 417, that the evaluation of the coefficient of form ultimately must depend upon measurements applied to an entire trajectory, it is a natural query at this point how we are to perform the

* Helmert's formula is generally accepted in ballistics. Omitting terms pertaining to height above sea level and to local abnormalities in density of the earth's structure, the formula is $g = (32.172 - 0.0853 \cos 2l)$ f.s.s., l being the latitude. The term for change of g with altitude gives this change to be $(-)$.003 f.s.s. per 1000 feet above sea level. Ref. Chapter II, *New Methods in Exterior Ballistics*, Moulton; also pp. VIII and IX, *Exterior Ballistic Tables Based on Numerical Integration*, Vol. I, 1924 (U. S. Army); also *T.S. No. 148*, April, 1921, (U. S. Army).

above solution, which requires knowledge of the value of i , if the value of i in turn depends upon the solution itself. The situation is such that, having actually fired a gun under a given set of conditions (i.e., with ϕ , V , δ , w , and d all carefully measured), we measure the resulting range X , and the problem then is to find the value of i which, when combined with the identical ϕ , V , δ , w , and d used in the firing, will reproduce the identical X actually measured. One way of accomplishing this would be to proceed by making a solution with an estimated value of i and, by comparing the resulting computed range with the required value, making a better estimate of i and a further solution, and so on—in short, by a process of successive approximations. Even with absolutely no prior knowledge as to the value of i we might thus eventually arrive at its value, although at the expense of great labor.

The equivalent of the above is indeed what is resorted to, although after a certain amount of preliminary work has been done and recorded in advantageous form, the remaining process for any given situation is reduced to exceedingly simple terms. Solutions are made for a great many trajectories, embracing all values of ϕ , V , and C that are likely to occur in practice. The computed values of the terminal elements X , ω , T , v_u , and of y_u , are then tabulated against the arguments ϕ , V , and C in such manner that the values of these elements may be found for any combination of values of the arguments. Now if a gun is fired at a known angle of departure ϕ and the initial velocity V of the shot is measured, and if the resulting range X also is measured, then it is a simple matter to find in our tables what value of C corresponds to the given values of ϕ , V , and X . Having thus determined the value of C , and having also measured δ , w , and d , it then remains but to find the value of i which, in combination with the given δ , w , and d , satisfies the given value of C (i.e., we solve for i by means of (406) transposed to the

form $i = \frac{w}{\delta C d^2}$). Further details of this process will be dealt with in the next

chapter, and the principal features of the proving-ground phases involved therein will be discussed in Chapter 8.

625. It would appear, without giving due thought to the matter, that if it is necessary to determine X in the first place in order to determine a value of i , it should be unnecessary to do anything further at all than to record the value of X for future reference—for after all i serves no better purpose than to permit the already known value of X to be reproduced by computation. This would be true if by experimental firing alone we could measure, not only X but all other required elements (ω , T , v_u , y_u), for all probable combinations of ϕ , V , δ , w , and d . This is altogether impracticable, not only because of the enormous amount of firing* that it would involve but also because it is practically impossible to measure such elements as ω , v_u , and y_u . Moreover, it is necessary also to know the values of elements at various points of the trajectory other than the point of fall, not only for antiaircraft work but also in order that corrections may be deduced for influences that vary in amount for different portions of the trajectory, as for example winds which vary at different altitudes (as they usually do).

* Such firing takes a great deal of time and is very expensive, since it involves not only the direct cost of guns and ammunition but also very considerable overhead costs for operating personnel, triangulation parties, patrol vessels, etc. The cost of only a few experimental shots from a major-caliber gun is sufficient to pay the annual wage of an expert computer.

It is true, strictly speaking, that a different value of i may be required for each different combination of ϕ , V , δ , w , and d , but it is also true that i varies slowly enough to permit interpolation between values determined experimentally for fairly widely separated combinations of the above elements. From experience it is found that experimental firing at relatively few angles of departure is sufficient for the construction of an entire range table for a given V and C , and for ordinary variations in the latter (such as variations in V due to erosion and powder temperature, and in C due to non-standard δ).

626. All that has been said with respect to the determination of the value of i for the type of solution illustrated in this chapter, applies equally well to any other type of solution of a trajectory. In Siacci's Method also, it is necessary to make what amounts to a solution backwards from a measured X and thus to find the value of i which satisfies the known value of X . And in Siacci's Method the process of making this backward solution is similar in nature to that outlined above for the numerical integration method, although it is somewhat less direct.* It should be apparent, then, that Siacci's Method or any other method may be forced, by determination of the appropriate i , to reproduce the actually measured X , just as the numerical integration was forced so to do. Also, it should be apparent that the value of i determined by a process of this character will, for the same set of conditions, vary according to the method of solution in which it is to be used. For we now find that, in addition to having to account for the many imperfectly known influences already discussed in articles 413-417, i has to account also for the differences in the degrees of approximation of the various methods of solution.

The fact that, by determination of the appropriate value of i , Siacci's Method can be made to produce just as correct a value of the range as can be gotten by the more laborious method of numerical integration, does not alter the situation that the latter affords an appreciably more accurate determination for points other than the point of fall. Siacci's Method holds a very decided advantage over the numerical integration method for finding terminal elements of trajectories of limited curvature; it does not, however, adapt itself satisfactorily to accurate determination of elements at intermediate points, for reasons already outlined in article 514.

627. It must be borne in mind, however, that the advantages as to accuracy that we have ascribed to the numerical integration method pertain entirely to the process of solution and do not overcome the difficulties arising from imperfect knowledge of physical values involved in the solution, except insofar as they afford a better separation of the effects of the latter from the effects of inaccuracies in the process of solution itself. The numerical integration process may be said to afford accuracy in the solution for any point of a trajectory up to the limits of accuracy within which the physical data pertaining to that point are known. In the particular case of the terminal point, the sum total effect of inaccuracies of all kinds, including those pertaining to physical data, can be wiped out by determining i as has already been described above. For intermediate points we depend on the assumption that i remains the same for all points in the trajectory, although this is not necessarily true.

By using the numerical integration process of solution we afford all intermediate points the same treatment as we do the point of fall, and thus eliminate

* See Appendix A.

changes in i which are incident to varying degrees of approximation for various portions of the same trajectory (such as are inherent in Siacci's Method). But in so doing we do not eliminate the possibility that i may vary along the trajectory due to physical causes. Thus the i determined for the point of fall includes, among other things, the effect of obliquity of the projectile as an average for the whole trajectory (art. 417); the average obliquity for the portion of the trajectory up to some intermediate point may, however, differ from the average for the whole trajectory, and the i for the intermediate point accordingly may differ from the i for the whole trajectory. More particularly it is probable that the assumed retardation function (such as the G -function) does not fit the given case exactly. The i determined for the point of fall wipes out the accumulated error, for the whole trajectory, due to the inaccuracy of the assumed retardation function. Any difference between the error accumulated up to an intermediate point and that accumulated up to the point of fall requires an adjustment of i for the intermediate point.

In the absence of explicit knowledge of each and every particle of physical data pertaining to each and every point in the trajectory, it becomes necessary to determine i for each point in a manner similar to that already described for the point of fall. Since the latter is generally impracticable, the accuracy of solutions for intermediate points is always limited by our knowledge of the necessary physical data.

628. At the Aberdeen Proving Ground recently, in the firing of .30 caliber and .50 caliber machine guns for which trajectories had been computed according to the G -function and with values of i that satisfied observed ranges, it was observed that the computed trajectories failed materially to agree at intermediate points with the observed trajectories (as indicated by tracers). Dr. Hedrick of the Aberdeen Proving Ground then invented a purely empirical method of varying i along the trajectory to secure a more satisfactory agreement. He found, in this case, that it was necessary to change i from point to point until a certain portion of the trajectory had been computed, and that thereafter a constant i could be used for the remainder of the trajectory. The difficulty in this case was ascribed to failure of the G -function to apply with sufficient accuracy to the projectile in use, and particularly so at certain velocities.

The U. S. Army has in recent years engaged in extensive research work for the purpose of establishing better retardation functions, and these efforts have materialized in the direction of establishing a number of such functions, a different one for each of several types of projectile. As better retardation data become available the advantages of accuracy inherent in the numerical integration method of solution will be gained more fully.

629. The use of ballistic tables has already been touched upon in article 624. Such tables not only constitute the practical basis for determining the value of i , but they are a great convenience for the computation of range tables. Nearly all methods of solving the trajectory that have been developed in the past have given rise to ballistic tables in which the results of the more tedious operations have been tabulated against convenient arguments. The nature of the ballistic tables pertaining to Siacci's Method has already been touched upon in article 511. The ballistic tables of the numerical integration method in its modern form afford a more direct approach to the ultimate, practical application to the construction of range tables than has been the case with ballistic tables of the various approximation methods (such as Siacci's). The greater labor involved in the basic computations according to the numerical integration method is therefore com-

compensated for in the simpler process remaining for its practical application to the preparation of range tables.

In addition to the various labor-saving devices that may be developed in the computational process itself, it is noteworthy that the computation of one long trajectory by numerical integration may be made to yield data for many shorter trajectories included within the limits of the long trajectory. This will be apparent from a study of Figure 13. Let us assume that the trajectory OSH of Figure 13 represents the trajectory for the conditions stated in article 611, and that the values of t , v , θ , x , and y have been recorded for the various intervals used in the solution. We can then find the values of all the elements at any point (by interpolation, if neces-

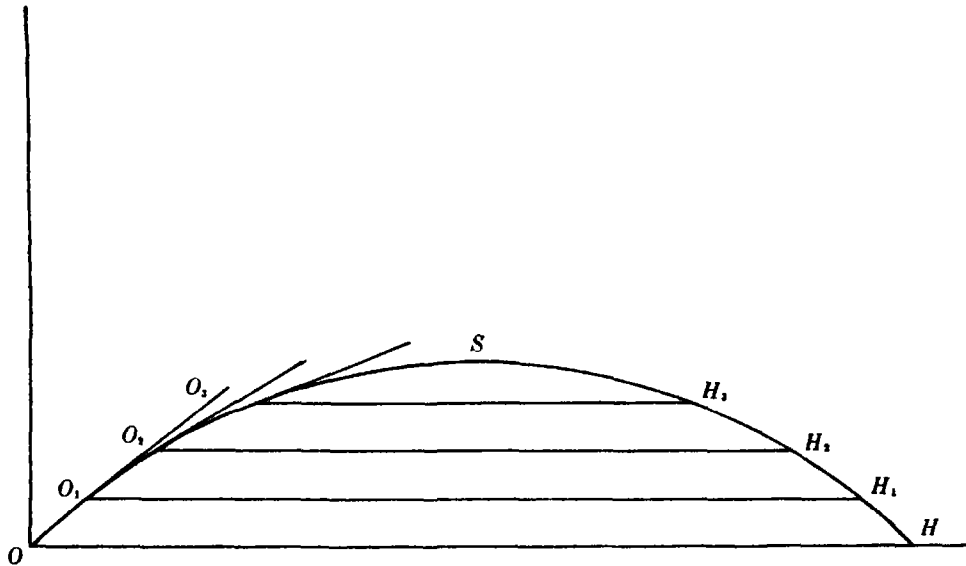


FIGURE 13

sary), as for example at the point O_1 , which we will assume to be the point where $\theta = 25^\circ$. For this point O_1 , where $\theta = 25^\circ$, we can then define t , v , x , and y . We also can find in the descending branch the point H_1 which lies in the same horizontal plane as O_1 ; this will be the point where y has the same value that y has at the point O_1 . Also, we can find t , v , θ , and x for the point H_1 . We now have sufficient data for the trajectory O_1SH_1 , which is the trajectory for the conditions $\phi = \theta = 25^\circ$, $V = v$ (at O_1), $v_\omega = v$ (at H_1), and $\omega = \theta$ (at H_1). Also, X for the trajectory O_1SH_1 equals x (at H_1) minus x (at O_1); T equals t (at H_1) minus t (at O_1); and y , equals y , for the trajectory OSH minus y (at O_1 and H_1). Numerous other trajectories, such as O_2SH_2 , O_3SH_3 , etc., can be determined similarly.

630. Since atmospheric density varies with altitude we must, however, adjust the value of C to the zero level of each trajectory cut from the principal trajectory. This is done quite simply by letting C for the zero level of any included trajectory

equal $\frac{C}{H_v}$ of the principal trajectory for the level of the latter which corresponds

to the zero level of the former. For example, let us assume that at O_1 the altitude is $y = 8000$ feet, for which $\log H_v = 9.89024 - 10$, and that $\log C = 1.12865$. It is

immaterial whether we use, at point O_1 , $E = \frac{G_0 \times H_0}{C}$ with $\log H_0$ and $\log C$ having

the values stated above, or $E = \frac{G_0 \times H_0}{C}$ with H_0 equal to unity and $\log C$

having the value $(1.12865) - (9.89024 - 10) = 1.23841$. That is, the value $\log C = 1.23841$ corresponds to zero altitude for the trajectory O_1SH_1 , and in tabulating the elements for this trajectory we must tabulate them not only against the ϕ and V found as already noted above, but also against the $\log C$ here found.

This process is valid because H_0 is an exponential function, and we can verify this by a numerical example as follows. It should be possible to compute a point lying beyond O_1 with identical results whether we work from the origin and use $\log C = 1.12865$ and $\log H_0$ as measured with respect to the origin, or whether we work from the point O_1 and use $\log C = 1.23841$ and $\log H_0$ as measured with respect to the point O_1 . Let us suppose, then, that the point where $y = 9000$ feet is to be found. Working from the origin we have $\log C = 1.12865$ and $\log H_0 = 9.87652 - 10$ (using $y = 9000$ feet); working from point O_1 we have $\log C = 1.23841$ and $\log H_0 = 9.98628 - 10$ (using $y = 1000$ feet, i.e., 1000 feet higher than 8000 feet, which is the level at O_1). Either of these two sets of $\log C$ and $\log H_0$ results in exactly the same value of E when substituted in (601).

It appears, therefore, that each set of ϕ , V , and C assumed for the computation of a trajectory by numerical integration yields, in addition to the principal trajectory defined by these arguments, a number of additional trajectories defined by an assortment of sets of ϕ , V , and C . The arrangement of these additional solutions against convenient tabular intervals of the arguments involves much painstaking interpolation, but nevertheless it affords an economy of labor in comparison to the very tedious step-by-step numerical integration for all of these additional sets of arguments.*

EXERCISES

1. Using the same data and process as stated for the example given in article 604, compute the distances covered by the projectile in the intervals while its velocity was being reduced, (a) from 1800 f.s. to 1790 f.s., and (b) from 1790 f.s. to 1780 f.s.

Answer (a) 42.0 feet (b) 42.2 feet.

2. Using the same data and process as stated for the example given in article 604, compute the distance covered by the projectile while its velocity was being reduced from 1820 f.s. to 1810 f.s., making the computation in five 2 f.s. intervals.

Answer 41.7 feet.

3. Using the same data and process as stated for the example in article 608, compute the distances covered by the projectile during the following intervals, and remaining velocities at the ends of these intervals: (a) 0.2 sec. to 0.3 sec. (b) 0.3 sec. to 0.4 sec. (c) 0.4 sec. to 0.5 sec.

Answer (a) 169.8 feet (b) 166.0 feet (c) 162.4 feet
1678.8 f.s. 1641.6 f.s. 1605.9 f.s.

* It affords, also, a means of tabulating not only terminal points but many intermediate points, without increasing the number of arguments. Volume I of the U. S. Army's *Exterior Ballistic Tables Based on Numerical Integration* is based on this principle. Further details may be found in the introductory pages of these tables.

4. Using the same data and process as stated for the example in article 608, compute the distance covered by the projectile in the first 0.1 sec., making the computation in two intervals of 0.05 sec. each.

Answer 177.9 feet.

5. Using the same data and process as stated for the example in article 608, compute the distance covered by the projectile in the first 0.5 sec., and remaining velocity at the end of that time, making the computation in one interval. (Compare with results given in last paragraph of article 608.)

Answer 851.3 feet; 1605.3 f.s.

6. Continue the problem stated in article 611 to the point $t=6$ seconds, making the computations for one $\frac{1}{2}$ -second interval.

ANSWERS

t	v	θ	x	x'	x''	y	y'	y''
2	2463.2	28°44'.6	4409.1	2159.7	42.59	2482.2	1183.9	55.52

CHAPTER 7

THE CONSTRUCTION AND USE OF BALLISTIC TABLES BASED ON NUMERICAL INTEGRATION.

701. The principles involved in the computation of trajectories by numerical integration have been covered in the preceding chapter, and the need for ballistic tables in connection with the practical applications of this method also has been touched upon. We shall now examine in greater detail the character of these tables and the features entering into their use.

It is to be noted, first of all, that the physical data which define a trajectory are the velocity-retardation law, the altitude-density law, the value of the gravitational constant g , and the values of ϕ , V , and C .^{*} Having assumed a definite velocity-retardation law, such as the G -function, a definite altitude-density law, such as the H -function, and a definite value of g , the remaining parameters which define a trajectory are ϕ , V , and C . In modern ballistic tables based on numerical integration, ϕ , V , and C are therefore the entering arguments. It should not be overlooked, however, that the assumed velocity-retardation law, altitude-density law, and value of g , are inherent in such tables, and that the tables are strictly applicable only to cases for which these assumptions are correct. Minor variations from these assumptions can be accounted for very conveniently, and with sufficient accuracy for practical purposes, through the medium of the coefficient i (as outlined in articles 623, 627, and 628). A material variation from these assumptions, as for example a materially different velocity-retardation function, requires a new set of tables, or some process which gives the equivalent thereof.

702. The French were among the first to issue comprehensive ballistic tables based on the modern application of numerical integration. In September, 1918, the *Ministère de l'Armement* issued tables which had been prepared by the *Commission Artillerie Lourde sur Voie Ferrée* (freely translated, Heavy Railway Artillery Commission); in 1921, the tables were extended to include a wider range of the arguments and some additional elements. These tables were adopted in this country shortly after their first appearance, and have come to be known as the *A.L.V.F. Tables*. The U. S. Navy has continued up until the present time to use these tables extensively in connection with the construction of its range tables for angles of departure above 15° , and we shall, accordingly, devote our present study principally to them.

The *A.L.V.F. Tables* are based on the velocity-retardation relation expressed by the Gâvre function, differing but little from the G -function (art. 410), and on an altitude-density relation of the same character as the H -function but differing very slightly from the latter. Numerical integration of the differential equations

^{*} Certain other factors, such as the forces introduced by the spin of the projectile and the rotation of the earth, as well as wind and motion of gun, might be included here. However, the effects of these influences are always dealt with separately as corrections to the trajectory, as will be seen presently.

derived in Chapter 5 (art. 503), with θ taken as the independent variable, was used in computing the tables.

In 1919, the U. S. War Department issued a translation of the first (1918) edition of the A.L.V.F. Tables. This translation involved chiefly the conversion of the argument C from the French standards to those used in the United States†; metric units for ranges and velocities, as used by the French, were retained in the American translation. Several extracts from the American translation of the 1918 A.L.V.F. Tables are given in Table VI, *Range and Ballistic Tables, 1935*. Sample pages covering the illustrative examples that will be given in this chapter are appended at the end of this chapter.

703. The general arrangement of data in the A.L.V.F. Tables is illustrated in the extracts appended to this chapter, which represent four pages of these tables, $\phi = 25^\circ$ being the page argument and V and C the vertical and horizontal arguments, respectively, for each page. A triple-entry system is obtained by having separate pages for the several tabular values of the argument ϕ , each page having the additional arguments V and C . (Compare with the *Azimuth Tables*, in which latitude is the page argument, and hour angle and declination are the vertical and horizontal arguments, respectively, for each page.) In order to accommodate a wide range of values of V and C and sufficiently small tabular intervals of each without unduly increasing the size of the page, a number of pages is required for each value of ϕ . In the American edition of the A.L.V.F. Tables each page includes one-half of the total range of values (300 m.s. to 900 m.s.) of the argument V , and one-fourth of the total range of values (about 3 to about 15) of the argument C . Also, only one element is tabulated on each page. For each tabular angle of departure there are, therefore, eight pages for each of the tabulated elements (X , T , ω , and v_s), or thirty-two pages for each tabular value of ϕ . A set of pages as just described is given for each 5° interval in ϕ from 15° to 40° inclusive. The untranslated 1921 edition includes $\phi = 45^\circ$, V up to 1200 m.s., and y_s in addition to the elements already noted.

The tabular interval in the argument V is uniform throughout the tables and is 10 m.s. For the argument C , both C itself and its logarithm are given. Since interpolation with respect to $\log C$ is both more convenient and more accurate than interpolation with respect to C itself, a uniform tabular interval of .020 in $\log C$ is used; this corresponds to a tabular interval of about 4.7% with respect to C itself. The odd tabular values of C are due to the conversion from the French C of the original tables.

704. The units used in the A.L.V.F. Tables are as follows:

- (a) *Initial velocity* (V) is tabulated in meters per second.
- (b) *Striking velocity* (v_s) is tabulated in decimeters per second; meters per second are obtained by pointing off one place from the right.

† This conversion accounts principally for the difference between French and American practices as to the manner of defining the coefficient of form and of setting up the ballistic coefficient. Further details may be found in *T.S. No. 148* (U. S. Army, 1921); also p. V, *A.L.V.F. Ballistic Tables* (U. S. Army, 1919), and p. 52, *Report on Computation of Firing Tables for the U. S. Army*, by H. P. Hitchcock.

- Units used in A.L.V.F. Tables
- (c) *Range* (X) is tabulated in meters.
 - (d) *Time of flight* (T) is tabulated in hundredths of a second; seconds are obtained by pointing off two places from the right.
 - (e) *Angle of fall* (ω) is tabulated in half-minutes of arc; degrees and minutes are obtained by dividing by 120 (for degrees) and the remainder by 2 (for minutes).

For example, let us look up the values of the various elements corresponding to the arguments $\phi = 25^\circ$, $V = 670$ m.s., and $\text{Log } C = 1.055$ (these may be found in the extracts appended to this chapter). For range we find $X = 20,875$ meters. For time of flight we find $T = 49.20$ seconds. For angle of fall we find 4244 half-minutes; dividing 4244 by 120 to get whole degrees, and the remainder by 2 to get minutes, we have $\omega = 35^\circ 22'$. For terminal velocity (or striking velocity) we find $v_s = 379.4$ m.s.

705. For converting metric units to English units, and vice versa, we shall use the following relations.*

- 1 meter = 3.2808 feet (log 0.51598)
- 1 foot = .30480 meter (log 9.48402 - 10)
- 1 meter = 1.0936 yard (log 0.03886)
- 1 yard = .91440 meter (log 9.96114 - 10)

It is not necessary that the ballistic coefficient C be found in metric units, since it is a ratio and hence independent of units. The necessary conversion due to the difference between the French ballistic coefficient and our own C has been made in preparing the translation from the original French tables; the value of $\text{Log } C$ for use with the translated edition is to be found exactly as explained heretofore.

706. The tabular intervals of the arguments V and C in the A.L.V.F. Tables are small enough to make second differences negligible for purposes of interpolation. Ordinary linear interpolation is therefore used with respect to V and C . In order to facilitate interpolation with respect to these arguments, differences between adjacent tabular values are recorded alongside the latter in the body of the table. Thus immediately to the right of each value tabulated in the body of the table will be found the difference between the latter and the value next to the right of it, i.e., the difference with respect to adjacent tabular values of the argument C . Similarly, immediately below the difference with respect to adjacent tabular values of C will be found the difference with respect to adjacent tabular values of V . For example, on the page headed $\phi = 25^\circ$ *Range*, with $V = 790$ m.s. and $\text{Log } C = 1.095$, we find $X = 27,529$; this is the range in meters corresponding to $\phi = 25^\circ$, $V = 790$ m.s., and $\text{Log } C = 1.095$. Immediately to the right of 27,529 we find the number 540, which is the difference between the values of X for $\text{Log } C = 1.095$ and for $\text{Log } C = 1.115$ (28,069 - 27,529), the arguments ϕ and V having remained the same. Immediately below the number 540 we find the number 516, which is the difference between the values of X for $V = 790$ m.s. and for $V = 800$ m.s. (28,045 - 27,529), the arguments ϕ and $\text{Log } C$ having remained the same. The arrangement is quite obvious, since the differences with respect to the argument C appear to the right of the tabular values to which they pertain, this being the direction in which the argument C increases; while the differences with respect to the argument V appear below the tabular values to which they pertain, this being the direction in which

* In the United States the value of the meter is legalized at 39.37 inches; in Great Britain and France, 39.37079 inches are considered to be the equivalent of one meter.

the argument V increases. Caution must be exercised, however, in the matter of the signs of these differences. The signs are not given in the tabulation but may be determined readily by inspection. It will be noted that X , T , and v_∞ always increase as either V or C increases, while ω increases as V increases but decreases as C increases.

The following examples cover the various cases of interpolation with respect to V and C . The portion of the A.L.V.F. Tables required in connection with these examples is given in the extracts appended to this chapter.

Single interpolation with respect to C **707. I. Given:** $\phi = 25^\circ$, $V = 790$ m.s., $\text{Log } C = 1.12707$.
Find: Range, time of flight, angle of fall and terminal velocity (in metric units).

We find $V = 790$ m.s. is a tabular value, hence this is a case of single interpolation. Then we find, in the table headed $\phi = 25^\circ$ **Range**,

For $V = 790$ and $\text{Log } C = 1.115$, $X = 28,069$

For $V = 790$ and $\text{Log } C = 1.135$, $X = 28,607$

The difference between the two values of X is 538, and is given in the difference column next to 28,069. Then we have, for $V = 790$ and $\text{Log } C = 1.12707$,

$$X = 28,069 + \frac{.01207}{.020} \times 538 = \underline{28,394 \text{ m}}$$

From the table headed $\phi = 25^\circ$ **Time of Flight**, we have,

For $V = 790$ and $\text{Log } C = 1.115$, $T = 5758$

For $V = 790$ and $\text{Log } C = 1.135$, $T = 5796$

The difference between the two values of T is 38, and is given in the difference column next to 5758. Then we have, for $V = 790$ and $\text{Log } C = 1.12707$,

$$T = 5758 + \frac{.01207}{.020} \times 38 = 5781 \text{ or } \underline{57.81 \text{ seconds}}$$

Note that the pointing off of decimal places may be done most conveniently after the interpolation has been completed.

From the table headed $\phi = 25^\circ$ **Angle of Fall**, we have,

For $V = 790$ and $\text{Log } C = 1.115$, $\omega = 4379$

For $V = 790$ and $\text{Log } C = 1.135$, $\omega = 4335$

The difference between the two values of ω is 44, and is given in the difference column next to 4379. Then we have, for $V = 790$ and $\text{Log } C = 1.12707$,

$$\omega = 4379 - \frac{.01207}{.020} \times 44 = 4352$$

whence we have

$$4352/120 = \underline{36^\circ 16'}$$

Here again it is convenient to complete the interpolation before converting any values into degrees and minutes.

From the table headed $\phi = 25^\circ$ **Terminal Velocity**, we have,

For $V = 790$ and $\text{Log } C = 1.115$, $v_\infty = 4223$

For $V = 790$ and $\text{Log } C = 1.135$, $v_\infty = 4296$

The difference between the two values of v_∞ is 73, and is given in the difference column next to 4223. Then we have, for $V = 790$ and $\text{Log } C = 1.12707$,

$$v_w = 4223 + \frac{.01207}{.020} \times 73 = 4267 \text{ or } \underline{426.7 \text{ m.s.}}$$

It will be observed that it is not necessary to write the data for each of the sets of values ($V=790-\text{Log } C=1.115$ and $V=790-\text{Log } C=1.135$), since the difference column gives us the differences to be used in interpolating. In this first example the data for the two sets of values have been put down in order to make the illustration complete, and in order that there may be no confusion as to the use of the quantities from the difference columns. Normally the entire procedure reduces to the following:

The interpolating ratio is,

$$\frac{1.12707 - 1.115}{.020} = \frac{.01207}{.020} = .604$$

whence,

$$\begin{aligned} X &= 28,069 + .604 \times 538 = \dots\dots\dots 28,394 \text{ m.} \\ T &= 5758 + .604 \times 38 = 5781 \text{ or } \dots\dots\dots 57.81 \text{ sec.} \\ \omega &= 4379 - .604 \times 44 = 4352 \text{ or } \dots\dots\dots 36^\circ 16' \\ v_w &= 4223 + .604 \times 73 = 4267 \text{ or } \dots\dots\dots \underline{426.7 \text{ m.s.}} \end{aligned}$$

Single
interpolation
with
respect to V

708. II. Given: $\phi = 25^\circ$, $V = 792.5 \text{ m.s.}$, $\text{Log } C = 1.11500$.

Find: Range, time of flight, angle of fall and terminal velocity (metric units).

In this case we find that $\text{Log } C = 1.11500$ is a tabular value, hence this also is a case of single interpolation. Remembering that the differences with respect to V appear below those with respect to C , we will proceed directly to the abbreviated process just illustrated in the last example.

The interpolating ratio is,

$$\frac{792.5 - 790}{10} = .25 \text{ or } \frac{1}{4}$$

whence,

$$\begin{aligned} X &= 28,069 + \frac{1}{4} \times 532 = \dots\dots\dots 28,202 \text{ m.} \\ T &= 5758 + \frac{1}{4} \times 61 = 5773 \text{ or } \dots\dots\dots 57.73 \text{ sec.} \\ \omega &= 4379 + \frac{1}{4} \times 17 = 4383 \text{ or } \dots\dots\dots 36^\circ 32' \\ v_w &= 4223 + \frac{1}{4} \times 21 = 4228 \text{ or } \dots\dots\dots \underline{422.8 \text{ m.s.}} \end{aligned}$$

Double
interpolation

709. III. Given: $\phi = 25^\circ$, $V = 792.5 \text{ m.s.}$, $\text{Log } C = 1.12707$.

Find: Range, time of flight, angle of fall, and terminal velocity (metric units).

Neither V nor C are tabular values, hence this is a case of double interpolation. We shall proceed by making the interpolation for C for the velocities next lower and next higher to the given velocity, and then interpolating between these values for the given velocity.

For $\text{Log } C$ the interpolating ratio is,

$$\frac{1.12707 - 1.115}{.020} = .604$$

For V the interpolating ratio is,

$$\frac{792.5 - 790}{10} = \frac{1}{4}$$

Then we have, interpolating first for Log C and then for V ,

$$\text{For } V=790, \quad X=28,069+.604 \times 538=28,394$$

$$\text{For } V=800, \quad X=28,601+.604 \times 554=28,936$$

$$\begin{aligned} \text{For } V=792.5, \quad X &= 28,394 + \frac{1}{4}(28,936 - 28,394) \\ &= 28,394 + \frac{1}{4} \times 542 = 28,530 \text{ m.} \end{aligned}$$

$$\text{For } V=790, \quad T=5758+.604 \times 38=5781$$

$$\text{For } V=800, \quad T=5819+.604 \times 38=5842$$

$$\begin{aligned} \text{For } V=792.5, \quad T &= 5781 + \frac{1}{4}(5842 - 5781) \\ &= 5781 + \frac{1}{4} \times 61 = 5796 \text{ or } 57.96 \text{ sec.} \end{aligned}$$

$$\text{For } V=790, \quad \omega=4379-.604 \times 44=4352$$

$$\text{For } V=800, \quad \omega=4396-.604 \times 45=4369$$

$$\begin{aligned} \text{For } V=792.5, \quad \omega &= 4352 + \frac{1}{4}(4369 - 4352) \\ &= 4352 + \frac{1}{4} \times 17 = 4356 \text{ or } 36^\circ 18' \end{aligned}$$

$$\text{For } V=790, \quad v_w=4223+.604 \times 73=4267$$

$$\text{For } V=800, \quad v_w=4244+.604 \times 75=4289$$

$$\begin{aligned} \text{For } V=792.5, \quad v_w &= 4267 + \frac{1}{4}(4289 - 4267) \\ &= 4267 + \frac{1}{4} \times 22 = 4273 \text{ or } 427.3 \text{ m.s.} \end{aligned}$$

710. The above examples cover cases of entry into the A.L.V.F. Tables with the usual arguments, viz., ϕ , V , and C . Occasion arises also for determining

the value of the argument C that corresponds to given values of ϕ , V , and X (as noted in article 624). This constitutes indirect entry into the tables, which is illustrated by the following example.

Entry into A.L.V.F. Tables, with ϕ , V , and X as arguments to find Log C .

IV. Given: $\phi=25^\circ$, $V=791$ m.s., $X=28,908$ m.

Find: Log C .

Examining the table headed $\phi=25^\circ$ Range, we find that, in the vicinity of $V=790$, the value $X=28,908$ lies between the columns headed Log $C=1.135$ and Log $C=1.155$. We will first find X , for the given value of $V=791$, in each of these columns.

$$\text{For Log } C=1.135, \quad X=28,607+.1 \times 548=28,662$$

$$\text{For Log } C=1.155, \quad X=29,143+.1 \times 565=29,199$$

Both of the values of X we have found are for $V=791$; since our given value of $X=28,908$ lies between them, we may find the value of Log C corresponding to it by the usual interpolation process. We have,

$$\begin{aligned} \text{Log } C &= 1.135 + \frac{28,908 - 28,662}{29,199 - 28,662} \times (1.155 - 1.135) \\ &= 1.135 + \frac{246}{537} \times .020 = 1.14416 \end{aligned}$$

711. The tabular interval in ϕ in the A.L.V.F. tables is too great for linear interpolation with respect to this argument, but the required accuracy may be obtained by extending the interpolation to second differences. When an entire range table is being prepared, and the values of the various elements have been found for the several tabular values of ϕ (i.e., 15° , 20° , 25° , 30° , 35° , and 40°), the problem of interpolation for values of these elements between the tabular values of ϕ can be handled about

Interpolation with respect to ϕ in A.L.V.F. Tables

as satisfactorily by graphical methods as by interpolation by formulas involving the higher differences. In constructing U. S. Navy range tables from the A.L.V.F. Tables, the graphical process is used. The values of the several elements are found from the A.L.V.F. Tables for each 5° interval in ϕ from 15° to 40°, and are plotted against ϕ . Fair curves are passed through the plotted points and the values at any number of required intermediate points may then be picked from the curves. Regularity in the intermediate values so obtained is secured both by using an appropriately large scale for the plotting, and by adjusting the values picked from the curves so that the second differences run smoothly. This graphical process is practically the equivalent of interpolation with second differences, and is more convenient when a great many intermediate values are to be found.

Use of second differences in interpolating with respect to ϕ

712. For isolated cases involving the determination of elements for other than tabular values of ϕ , the usual methods of interpolation with second differences may be applied. The formula for interpolation with second differences is*

$$= f_0 + m\Delta_1 + \frac{m(m-1)}{2} \Delta_2 \quad (701)$$

in which f is the value of the element to be found; f_0 is the next lower tabular value of this element; Δ_1 and Δ_2 are, respectively, the first and second differences for this element with respect to the argument against which the interpolation is to be made; and m is the interpolating ratio with respect to the same argument, i.e., the proportion of its tabular interval for which the interpolation is to be made. Due consideration must be given to the signs of both the first and second differences; also, since m is always less than unity, it is to be noted that the quantity $(m-1)$ is always negative.

For illustration we shall use the following example, taking tabular values for V and C in order to confine the interpolation to the argument requiring the use of second differences.

Given: $\phi = 23\frac{1}{2}^\circ$, $V = 790$ m.s., $\text{Log } C = 1.13500$.

Find: Range (metric units).

In order to find the required first and second differences with respect to ϕ we must find X for three successive values of ϕ , and we shall therefore find X for $\phi = 20^\circ$, 25° , and 30° , using $V = 790$ m.s. and $\text{Log } C = 1.13500$ in each case. Tabulating these values of X , the differences are found readily as follows.

ϕ	$V = 790$ m.s.	$\text{Log } C = 1.135$	
	\overline{X}	Δ_1	Δ_2
20°	25,130		
		+3477	
25°	28,607		-603
		+2874	
30°	31,481		

The interpolating ratio is $\frac{3.5^\circ}{5^\circ}$, or $m = 0.7$; the value of X for the tabular value

$\phi = 20^\circ$ (which is the next lower to $23\frac{1}{2}^\circ$) gives $f_0 = 25,130$; the first difference corresponding to f_0 is $\Delta_1 = +3477$, and the second difference is $\Delta_2 = (-)603$. Sub-

* Ref. p. 115, Marks' *Mechanical Engineers' Handbook*, Second Edition.

stituting these values in (701) we have,

$$f(\text{or } X) = 25,130 + 0.7 \times 3477 + 0.105 \times 603 = 27,627 \text{ m.}$$

Had V and C not been tabular values, it would have been necessary to determine X for each of the three tabular values of ϕ by interpolation just as already illustrated in article 709. The remaining process would then have been reduced to that shown above. Interpolation with respect to ϕ for the elements T , ω , and v_ω involves nothing different than shown in the above example for X .

713. If it should be necessary to find the value of C corresponding to given values of X , V , and ϕ , with ϕ not a tabular value, we proceed as illustrated in the following example.

Given: $\phi = 23\frac{1}{2}^\circ$, $V = 790$ m.s., $X = 27,754$ m.

Find: The value of $\text{Log } C$ corresponding to the above values.

Indirect entry
into A.L.V.F.
Tables when
 ϕ is not a
tabular value

By trial and error we find that the required $\text{Log } C$ lies between the tabular values 1.135 and 1.155; hence we shall find X for each of these values of $\text{Log } C$, and with $\phi = 23\frac{1}{2}^\circ$ and $V = 790$ m.s. in each case. Then for

$V = 790 \text{ m.s.}$		$\text{Log } C = 1.135$	
ϕ	X	Δ_1	Δ_2
20°	25,130		
		+3477	
25°	28,607		-603
		+2874	
30°	31,481		

whence for $\phi = 23\frac{1}{2}^\circ$, $X = 27,627$ m. (as already found in the preceding example).
Also for

$V = 790 \text{ m.s.}$		$\text{Log } C = 1.155$	
ϕ	X	Δ_1	Δ_2
20°	25,570		
		+3573	
25°	29,143		-611
		+2962	
30°	32,105		

whence for $\phi = 23\frac{1}{2}^\circ$, $X = 25,570 + 0.7 \times 3573 + 0.105 \times 611 = 28,135$ m.

We now have two values of X as follows,

$\phi = 23\frac{1}{2}^\circ$, $V = 790$ m.s., $\text{Log } C = 1.135$, $X = 27,627$ m.

$\phi = 23\frac{1}{2}^\circ$, $V = 790$ m.s., $\text{Log } C = 1.155$, $X = 28,135$ m.

The given value $X = 27,754$ m. lies between these, and we interpolate for the corresponding $\text{Log } C$ just as in article 710, thus

$$\begin{aligned} \text{Log } C &= 1.135 + \frac{27,754 - 27,627}{28,135 - 27,627} \times .020 \\ &= 1.135 + \frac{127}{508} \times .020 = \underline{1.140} \end{aligned}$$

Had V not been a tabular value, the process merely would have been lengthened by the requirement of interpolating with respect to V for each of the adjacent values of $\text{Log } C$.

714. A few years after the French issued their A.L.V.F. Tables, the U. S. War Department commenced the preparation of a much more comprehensive set of tables of its own, under the title *Exterior Ballistic Tables Based on Numerical Integration*, to which we shall refer hereafter as the *War Department Tables*. These tables are based on numerical integration of the differential equations that have been derived in article 610, with t as the independent variable. The G -function and H -function, and $g=9.80$ m.s.s., are employed, and the process of solution is similar to that illustrated in articles 611-620 and further outlined in articles 621-622.

The War Department Tables, when completed, will include three volumes. Volume I, issued in 1924, is designed to afford a means of determining numerous points of any trajectory likely to be encountered in practice, and is based on the principles outlined in article 630. Volume II, issued in 1931, is required in connection with Volume I; it gives the values of the summital elements x , y , t , v , in terms of the arguments ϕ , V , and C . Volume III, at the time of this writing, is still in the process of preparation; it will give the values of the terminal elements X , T , ω , v_ω , in terms of the arguments ϕ , V , and C . Volume III of the War Department Tables is quite similar in purpose to the A.L.V.F. Tables, although the arrangement is somewhat different.

The scope of the War Department Tables is 0° to 90° in ϕ , 80 m.s. to 1000 m.s. in V , and 0.000 to 1.250 in $\log C$. The tabular intervals for V and C vary among the several volumes but are in all cases greater than in the A.L.V.F. Tables. The tabular interval for ϕ is 1° from $\phi=0^\circ$ to $\phi=30^\circ$, and 2° from $\phi=30^\circ$ to $\phi=90^\circ$, in all cases. The greater scope in values of ϕ and the smaller tabular intervals therein, as compared to the A.L.V.F. Tables, constitute important advantages over the latter. The War Department Tables are applicable to all angles of departure; furthermore, the tabular intervals for all of the arguments (ϕ , V , and C) are small enough to render linear interpolation with respect to any of them satisfactory for ordinary problems.* (While second differences with respect to V are not altogether negligible, the error that results from neglecting them rarely exceeds a very few units in the last significant figure retained and is of little importance, especially in connection with the finding of the maximum ordinate, to which our immediate use of these tables is to be confined.)

715. Since the American edition of the A.L.V.F. Tables does not include the maximum ordinate, we shall use for this element the appropriate portions of Volume II of the War Department Tables. Extracts of sufficient scope to cover all exercises dealt with in this text are given in Table VII, *Range and Ballistic Tables, 1935*. Sample pages covering the illustrative examples that will be given in this chapter are appended at the end of this chapter.

Referring to Table VII, it is to be noted that the most important difference

* It is the author's understanding, at the time of this writing, that when Volume III of the War Department Tables is issued it will be adopted by the U. S. Navy Department for the preparation of its range tables, replacing the Ingalls-Siacchi Method now in use for angles of departure up to 15° and the A.L.V.F. Tables now in use for angles of departure above 15° . Although Volumes I and II by themselves afford a complete solution for any required trajectory, the process involved in connection with the use of these volumes is rather laborious as compared to the methods now in use, except in the case of finding the maximum ordinate from Volume II. Any further explanation of the arrangement or use of Volume I would lead to a digression that is not warranted, since no application of this volume is to be used in this text (nor is it, indeed, used by the Navy Department). The limited use of Volume II that is to be made in this text affords ample illustration of the character of this volume, and also a fairly good illustration of the character of Volume III.

between these tables and Table VI (A.L.V.F. Tables) is that in the former the page argument is V , and the vertical argument is ϕ . With the additional exception that the difference columns are not given in Table VII, the arrangement of the two tables is very similar. Further points to be noted in connection with Table VII are that it is in metric units, that the tabular interval in V is 40 m.s., and that the tabular interval in $\text{Log } C$ is .050.

716. Interpolation in Table VII offers no new problems. It is to be noted that interpolation with respect to the page argument V will ordinarily be required. However, this does not involve the complications met in interpage interpolation in the A.L.V.F. Tables, since in Table VII linear interpolation may be used between pages (i.e., with respect to V), as well as on each page (i.e., with respect to ϕ and C).

For illustration of the use of Table VII we shall take the same data as given in the example in article 709.

Given: $\phi = 25^\circ$, $V = 792.5$ m.s., $\text{Log } C = 1.12707$.

Use of the War
Department
Tables for
finding the
maximum
ordinate

Find: Maximum ordinate (metric units).

The given V lies between the tabular values 760 m.s. and 800 m.s.; we shall proceed by finding y_s for each of these values of V and with the given ϕ and $\text{Log } C$. Since ϕ is a tabular value this involves interpolation only with respect to $\text{Log } C$, for which the

interpolating ratio is

$$\frac{1.12707 - 1.10000}{.050} = .541.$$

Performing the required interpolation on the pages headed $V = 760$ m.s. and $V = 800$ m.s. we have, for $\phi = 25^\circ$ and $\text{Log } C = 1.12707$ in each case,

$$\text{For } V = 760 \text{ m.s., } y_s = 3815 + .541 \times 111 = 3875$$

$$\text{For } V = 800 \text{ m.s., } y_s = 4153 + .541 \times 127 = 4222$$

The interpolating ratio for V is

$$\frac{792.5 - 760}{40} = .812$$

whence we have

$$\text{For } V = 792.5 \text{ m.s., } y_s = 3875 + .812 \times 347 = \underline{4157\text{m}}$$

EXERCISES

1. Given: The values of ϕ , V , and $\text{Log } C$ (metric units).

Find: The values of X , T , ω , v_ω , and y_s (metric units).

	Given			Answers				
	ϕ	V (m.s.)	$\text{Log } C$	X (m.)	T (sec.)	ω	v_ω (m.s.)	y_s (m.)
A	15°	800.0	1.11500	21,067	37.30	21° 48'	449.1	1715
B	25	680.0	1.12500	22,633	50.89	34 33	400.8	3210
C	30	612.5	1.01500	19,560	52.60	40 22	363.9	3463
D	40	791.6	1.11590	34,828	85.30	52 00	453.9	9000
E	25	883.9	1.01600	29,686	60.74	39 50	403.3	4599

2. *Given:* The values of ϕ , V , and $\text{Log } C$ (English units).

Find: The values of X , T , ω , v_w , and y . (English units). (Use V to the nearest tenth of a m.s.)

	Given			Answers				
	ϕ	V (f.s.)	$\text{Log } C$	X (yds.)	T (sec.)	ω	v_w (f.s.)	y , (ft.)
A	15°	2625	1.11500	23,043	37.30	21° 48'	1473	5626
B	25	2231	1.12500	24,751	50.89	34 33	1315	10,532
C	30	2010	1.01500	21,401	52.62	40 22	1194	11,368
D	40	2597	1.11590	38,088	85 30	52 00	1489	29,527
E	25	2900	1.01600	32,465	60.74	39 50	1323	15,092

3. *Given:* The values of ϕ , V , and X (English units).

Find: The values of $\text{Log } C$ and ω . (Use V to the nearest tenth of a m s)

	Given			Answers	
	ϕ	V (f s)	X (yds.)	$\text{Log } C$	ω
A	15°	2625	23,043	1.11500	21° 48'
B	25	2231	24,752	1.12500	34 33
C	30	2010	21,401	1.01500	40 22
D	40	2597	38,088	1.11590	52 00
E	25	2900	32,465	1.01600	39 50

CHAPTER 8

THE DETERMINATION OF INITIAL VELOCITY AND COEFFICIENT OF FORM BY EXPERIMENTAL FIRING AT THE PROVING GROUND. THE DETERMINATION OF RANGE-TABLE VALUES OF RANGE, ANGLE OF DEPARTURE, ANGLE OF FALL, TIME OF FLIGHT, STRIKING VELOCITY, AND MAXIMUM ORDINATE (RANGE-TABLE COLUMNS 1, 2, 3, 4, 5, and 8).

801. Such tables as the A.L.V.F. Tables and War Department Tables are called *ballistic tables*, and they constitute a basis for the solution of trajectories corresponding to any combination of V and C that is likely to occur in practice.

The distinction between ballistic tables and range tables

In order that the data for a given gun may be available in convenient form, a *range table* is prepared for each gun. A range table is, first of all, a tabulation of elements of trajectories corresponding to the initial velocity and projectile of a particular gun, as compared to a ballistic table, which is applicable to many velocities and many projectiles. In this sense a ballistic table may be thought of as a master range table.* A range table has the additional feature of making available the data corresponding to the particular gun sufficiently in detail to render interpolation unnecessary in connection with ordinary service problems.

Although a range table is designed to serve for a particular gun having a specified initial velocity and projectile, it is necessary to provide for normal variations in these factors, as well as in others that affect the trajectory. In order to avoid the complications that would result from adopting a multiple-entry system to take care of such variations, it is the practice to tabulate in a range table the values of the desired elements under a standard set of conditions, and to provide therein also additional data by means of which the tabulated values may be adjusted for variations from the assumed standards. The standard conditions assumed for range-table values are,

- Standard conditions assumed for range-tables
- (a) That the projectile leaves the gun with the designed velocity.
 - (b) That the projectile has the designed weight and form.
 - (c) That the air at the surface is at the standard temperature (59°F.), pressure (29.53 inches of mercury), and saturation (78%), and that its variations aloft follow the assumed standard relations (art. 419-420).
 - (d) That there is no wind.
 - (e) That the gun is motionless.
 - (f) That the target is motionless. (Although motion of the target does not affect the trajectory itself, it enters into the problem of determining the trajectory that will reach the desired objective.)
 - (g) That the earth is motionless.
 - (h) That the gun and target are in the same horizontal plane.
 - (i) That the gun is warm.

* This is particularly true in the case of tables such as the A.L.V.F. Tables and War Department Tables, from which the desired elements are obtained directly. Some other ballistic tables, such as Ingalls' Tables (art. 511 and Appendix A), list not the elements themselves, but functions which are required for the solution of formulas for the elements.

802. We shall deal, in this chapter, with the problem of determining the values of range, angle of fall, time of flight, striking velocity, and maximum ordinate, corresponding to various angles of departure, that are tabulated in Columns 1, 2, 3, 4, 5, and 8 of the range table. All of these are elements for which values may be obtained by the processes of solution already outlined in foregoing chapters. Our immediate problem in the present chapter has to do with the application of the processes previously considered to the determination of values of these elements under the standard conditions assumed for a range table.

It is to be noted that the standard conditions stated above are in accord with the assumptions which were defined originally in article 213, and that no departure from the latter assumptions has been made in connection with the processes of solution and preparation of the ballistic tables which have been discussed in foregoing chapters. These ballistic tables therefore will yield directly values that correspond to the standards assumed for range tables. Insofar as the argument V is concerned, it is merely necessary to enter with the chosen standard value of V .

The determination of the argument C also is very simple, insofar as the factors δ , w , and d are concerned, for δ equals unity, and w and d are simply the designed weight and diameter of the projectile to which the range table is to apply. The remaining factor, i , which is contained in C , cannot be assigned a standard value, however, and must be evaluated by experimental firing.

The numerous considerations that enter into the value of the coefficient of form, i , have been discussed in detail in articles 413-417 and 624-626. All that has previously been said with respect to the value of i may be summarized by the statement that *the value assigned to i must be such as to cause agreement between the*

values of elements of the trajectory as determined by computation and as determined by experiment. Our problem thus requires, first of all, measurement of the range X which results from a firing of the given gun and projectile, and of the values of ϕ , V , δ , w , and d for that firing,—and determination of the value of i which, in combination

with the measured ϕ , V , δ , and w , and d , reproduces the measured X by whatever system of computation or tables we have chosen to use. Since the use of ballistic tables is an essential expedient in carrying through this process, as already noted in article 624, it is important that due consideration be given to all variations between the conditions under which the experimental firing is conducted and the standard conditions upon which the ballistic tables are based. We shall now examine briefly the principal features that enter into the experimental ranging of a gun and into the determination of the measured values of ϕ , V , δ , w , d , and X that are applicable to the ballistic tables; we may proceed thence to our more immediate problem of using these values to find i and of using the latter in the construction of the range table.

803. The experimental ranging of a gun at the proving ground, for the determination of range-table data, consists of the firing of several shots (from four to seven) at each of several angles of departure (8° , 15° , 25° , 35° , 45° , 15° , is a typical series), the gun being mounted in its regular service mount.

The following observations are made in connection with the firing.

- (a) The gun is laid by quadrant at exactly the required angle of departure before each shot.
- (b) The initial velocity for each shot fired at an angle of departure of 15° or less is determined from the velocity measured by oscillograph at a

short distance from the muzzle, and the initial velocities of the several shots fired at each of these angles of departure are averaged to give the initial velocities for these groups of shots. Since the rigging of screens or coils for velocity measurement is impracticable for angles of departure above about 15° , the initial velocities for the groups fired above 15° are determined by interpolation between the initial velocities actually measured for two groups at 15° , one preceding and one following the groups fired above 15° .* Due care having been taken to preserve uniformity as to weight and temperature of the powder charge throughout the firing, comparison of the measured velocities of the two 15° groups affords a good estimate of the loss of initial velocity due to erosion, in terms of rounds fired, which is the basis of the interpolation for the initial velocities of the groups fired above 15° .

- (c) Throughout the firing meteorological observations are made, both at the surface and aloft, for the determination of the atmospheric density factor and wind. For each group of shots fired at the same angle of departure, an average ballistic density and wind are determined from the observations taken during the firing of that group.
- (d) The point of fall of each shot is plotted by intersecting bearings taken at four observation stations. The distance from the gun to the point of fall is the uncorrected observed range. The perpendicular distance from the line of fire to the point of fall is the uncorrected observed lateral deviation. For each group of shots fired at the same angle of departure, average values of the uncorrected observed range and lateral deviation are determined from the individual values observed for the shots of that group. For use in connection with reduction of the observed range to the standard datum plane, the average height of the gun above tide level is recorded for each group. Also, the average weight of projectile for each group is recorded.

(e) Any shots fired while the gun is cold are excluded from the ranging data.

Some of the features outlined above will now be considered in somewhat greater detail.

804. It is to be noted that in laying the gun for the experimental ranging no allowance is made for the angle of jump. The angle of departure actually obtained therefore varies from the angle indicated by the quadrant by the amount of the angle of jump, and the observed range corresponds to the quadrant elevation plus or minus the angle of jump rather than to the quadrant elevation as actually laid. However, the angle of jump is an inherent part of the angle of departure and will be present in any subsequent firing of the gun as well as in the experimental range firing. Nothing is to be gained by attempting to separate the angle of jump from the angle of departure at the experimental ranging, as will appear from the following example. Let us suppose that a gun has been laid at the quadrant elevation 15° for experimental ranging, and that the angle of jump is known to be $+2'$; also that the resulting range, reduced to range-table standards, has been found to be 20,000 yards. The range 20,000 yards therefore corresponds actually to the angle of departure $15^\circ 02'$; if we should tabulate the latter value in the range table, we would then be obliged to tabulate also the angle of jump $+2'$, and to subtract this from $15^\circ 02'$ to obtain the 15° which we already know is required to

* It is expected that facilities for measuring velocities for shots fired at elevations up to at least 30° will become available in the near future.

give the range 20,000 yards. The identical result is obtained by tabulating 15° against 20,000 yards in the first place. The same applies in any case, since however we may determine the angle of jump and whatever we may find its value to be at the experimental ranging, we can do no better than to assume that this value will remain the same for the subsequent service firing of the gun at the same elevation.*

This method of avoiding separate consideration of the angle of jump obviously is a source of disagreement between computed and observed results, for it results in comparison of the observed range for one angle of departure with the computed range for a slightly different angle of departure. However, the angle of jump is small enough in any normal case to make this a feature of little importance even for theoretical considerations, and of no consequence at all in connection with our practical problem of constructing a range table. It is sufficient to note that the small discrepancies between observed and theoretical results that are attributable to this procedure are, like many others, accounted for by the coefficient of form.

805. What has been said above in connection with jump applies generally also to droop. Although the droop itself can be measured, its effect on the angle of departure cannot very well be separated from that due to the jump, for the droop may operate not only to cause the angle of departure to be modified slightly by the curvature of the bore, but also to cause a whip of the bore as the projectile travels through it. Whatever pains might be taken to evaluate these effects separately at the experimental ranging, we would still be obliged to assume identical effects for the subsequent service firings. The elimination of erratic or excessive droop or jump is a requirement of acceptable design of guns and their mounts.

It is to be noted, however, that in the service use of a gun the orientation of the sights by bore-sighting results in the use of the chord joining the centers of the breech and muzzle as the axis of the bore, whereas the laying of the gun by means of a quadrant, as is done at the proving ground, results in the use of the designed axis of the bore. A gun laid level by means of the axis of boresighting therefore actually has a slight elevation as compared to the same gun laid level by quadrant, the difference being caused by the droop (which lowers the muzzle end of the axis of boresighting). The effect of this is to cause guns in actual service to overshoot their range tables according to the amount of droop that existed at the time of boresighting. The deviation of the chord joining the centers of the breech and muzzle from the designed axis of the bore normally does not exceed a few minutes of arc.

806. The determination of initial velocity involves two principal features, viz.,

- | | |
|---|--|
| Experimental
determination
of initial
velocity | <p>(a) The measurement of the remaining velocity at a short distance from the muzzle.</p> <p>(b) The calculation of the initial velocity which corresponds to this remaining velocity, considering the velocity reduction that has occurred during the flight of the projectile from the muzzle to the point of measurement.</p> |
|---|--|

* The wording in the heading of Column 2 of our range tables is misleading in this connection, if interpreted literally. It is intended to be implied that the angle of jump has been considered in the tabulated values of angle of departure in the manner described above, and that no further corrections for jump are applicable. The correct interpretation is that the range tabulated in Column 1 corresponds to the angle of elevation tabulated in Column 2 plus the normal jump at that elevation.

The principles involved in the first of these features have already been discussed in article 403. The present practice at the Naval Proving Ground is to use an oscillograph for this purpose. The coils are so rigged that their heights may be adjusted for firing at angles of departure up to about 15° . Illustrations of the Naval Proving Ground layout for the measurement of velocities, and detailed descriptions of the apparatus used, are given in articles 773-781, *Naval Ordnance, 1933*.

The second of the above features involves, in effect, the solution of the trajectory between the muzzle and the point to which the measured remaining velocity pertains, which may be taken to be the point midway between the two coils. We have the values of x , y , and v for this point in the trajectory. Although the exact value of θ for this point is not directly available, no appreciable error will result from assuming that the short portion of the trajectory involved is a straight line, and that $\theta = \phi$. Knowing x , y , v , and θ for the given point, we can now work back to the origin and thus determine V . This solution might be made by numerical integration as outlined in Chapter 6, but for the conditions of the present problem the application of Siacci's Method affords a much simpler solution with negligible sacrifice of accuracy, and the latter is actually used.

807. The following formula, which has been derived in article 510, is directly applicable to the present problem.

$$x = C_s(S_u - S_v). \quad (519)$$

Calculation
of initial
velocity by
Siacci's
Method

According to this formula, x is the horizontal component of the distance covered by the projectile from the origin, where its velocity is V , to the point where its pseudo velocity is u ; S_v and S_u are the values of Siacci's space function for these velocities, and may be found from tables (art. 511). From formula (507) we have the relation

$$u = v \cos \theta \sec \phi \quad (801)$$

and since θ in this case varies but slightly from ϕ , it is sufficiently accurate for practical purposes in this connection to assume $u = v$, that is, that the pseudo velocity at the point of measurement equals the measured remaining velocity at that point. Transposing (519) we have

$$S_v = S_u - \frac{x}{C_s} \quad (802)$$

which may be solved by substituting for S_u the tabulated value of S for the velocity $u = v$, for x the horizontal component of the distance from the gun to the point midway between the oscillograph coils, and for C_s its value as given by formula (513).* The required value of V is then that which corresponds to the tabulated value of S_v found in this manner.†

Since the ballistic coefficient enters into the above determination, it is necessary to have a value of i before the initial velocity can be determined, although the initial velocity so determined is to enter eventually into the determination of i itself. Ordinarily a sufficiently accurate estimate of i for use in the determination of initial velocity is available from previous work. If the value of i eventually found for a given trajectory should vary materially from that assumed for the determination of

Value of i
for deter-
mination of
initial
velocity

* In actual practice, C is generally used in place of C_s , f_s and β being disregarded in view of their slight effect on the determination of V .

† These tables are given in the 1926 and 1930 editions of *Range and Ballistic Tables* but are omitted from the present edition due to the very limited references made to them in the present textbook.

the initial velocity used for that trajectory, the entire computation may be repeated, starting with a new determination of the initial velocity according to the better value of i established by the results of the first approximation. Similarly, if no previous knowledge of the required value of i should be available, a solution can be made by assuming an estimated value and making successive approximations until agreement is established within the desired limits of accuracy. This involves but little difficulty, considering the relatively small effect that variations in i have in the determination of the initial velocity.

808. It is worthy of note that the process outlined above, or any equivalent computational process for deducing the initial velocity from a velocity measured at a distance from the muzzle, does not in fact yield the true value of the projectile's velocity as it leaves the muzzle, but yields instead an initial velocity under the assumption that the projectile has suffered normal retardation from the instant of projection. Actually the acceleration of the projectile continues during a flight of as much as fifty yards from the muzzle, due to the action of the blast which escapes from the gun at high velocity and surrounds the projectile during the initial stage of its flight. The actual velocity at the muzzle therefore is not the greatest velocity attained by the projectile, nor is it the initial velocity in the sense in which the latter is employed in exterior ballistics.

The process used determines neither the true velocity at the muzzle nor the true maximum velocity actually attained by the projectile. The value of V so determined is a fictitious initial velocity which, according to the assumed velocity-retardation relation and for the actual distance covered by the projectile up to the point of measurement, satisfies the actual remaining velocity at that point. The only appreciable inaccuracy that results from this manner of handling the problem applies to values of the velocity itself for points short of the point of measurement. However, this is of no consequence in connection with the trajectory computation as a whole, since the initial velocity determined in the manner described has been chosen so as to cause agreement between computed and observed values at the point of measurement; in other words, it is the equivalent of the actual velocities between the muzzle and point of measurement. On the other hand, if means should be devised for obtaining an accurate measure of the velocity of the projectile at the muzzle, it nevertheless would remain necessary to make measurements also beyond the muzzle, at least up to the point where the projectile clears the blast from the gun and begins to encounter normal retardation. We would then have the choice of making a separate computation for the portion of the trajectory up to this point, or of deducing an equivalent initial velocity applicable to the usual system of computation. The latter is, in effect, what is accomplished by making a single measurement clear of the blast in the first place, and nothing is to be gained by attempting to make a close study of the velocity in the immediate vicinity of the muzzle, as far as our present problem is concerned.

809. The meteorological observations taken during the experimental ranging furnish data both for determining the value of δ that is to be used in connection with the other measured values from which i is to be deduced, and for determining corrections required to adjust the observed range and lateral deviation to the range-table condition of no wind. Accurate determination of air density and wind conditions encountered by the projectile throughout its flight requires measurements at several levels of altitude up to the height of the maximum ordinate. Aloft densities are determined

Nature of the experimentally determined value of initial velocity

Corrections for actual air density and wind

from measurements of air temperature and barometric pressure made by aircraft at approximately each 1000 feet of altitude, and are used for determining the ballistic density. Or, if aloft observations are not available, the ballistic density is determined from Table IV, as outlined in article 424. Aloft winds are measured by plotting the path of a small pilot balloon having a known ascensional rate, and are used for determining the ballistic wind.* (The method of finding the ballistic wind from aloft observations will be gone into further in articles 1114-1115).

810. In order to reduce the observed range to the conditions on which ballistic tables and range tables are based, it is necessary to apply to it corrections for the wind in the line of fire, and for the condition that the observed point of fall is not in the horizontal plane through the gun. The method of determining the correction for wind can be explained to better advantage after we have progressed further with other details of building up the range table, and it will be dealt with in Chapter 11; for the present we may accept that the necessary correction is available.

Two corrections are necessary to refer the observed point of fall to the horizontal plane through the gun, viz.,

- (a) A correction for the height of the gun above the surface of the earth (the tide level in this case).
- (b) A correction for the curvature of the earth.

Although these two corrections are due to entirely different causes, they are essentially similar in nature, as will be apparent from a study of Figure 14. In this figure the gun is assumed to be at O , at a height h_1 above the surface of the earth. OH is the horizontal plane through the gun, $O'H'$ is the horizontal plane

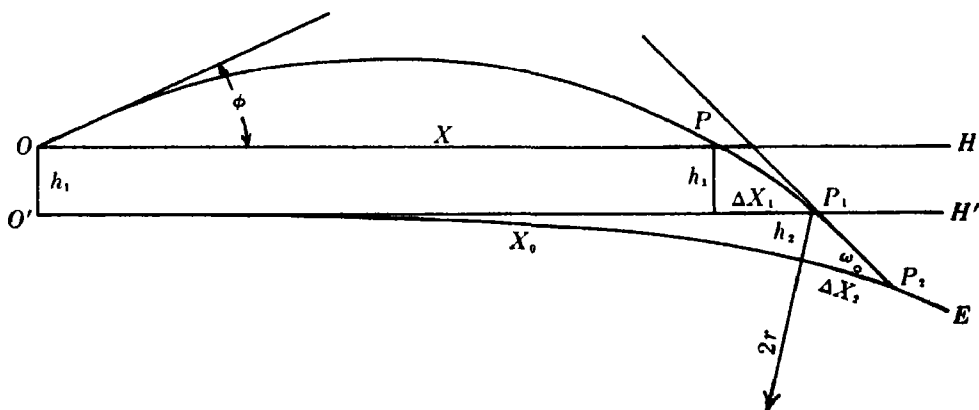


FIGURE 14

tangent to the earth at the point vertically beneath the gun, and $O'E$ is the curved surface of the earth. The observed point of fall is on the surface of the earth at P_2 , and $X_0 = O'P_2$ is the observed range. The required range X is the range to the point where the trajectory again intersects the horizontal plane through the gun, and it is equal to OP . It is sufficiently correct for practical purposes to assume that the observed range $X_0 = O'P_2$ exceeds the required range $X = OP$ by the sum of the two increments ΔX_1 and ΔX_2 , and that the latter are equal, respectively, to $h_1 \cot \omega_0$ and $h_2 \cot \omega_0$, ω_0 being the angle of fall for the observed range. Although the angles of fall at P_1 and P_2 differ slightly, the difference is small enough

* The method of measuring aloft winds by means of pilot balloons is described in detail in *Technical Regulations No. 1236-1* (U. S. War Department, 1934).

to be disregarded for the present purpose; a first approximation of the angle of fall based on the uncorrected observed range is sufficiently accurate for the determination of both of these corrections.

For the determination of ΔX_1 , the value h_1 is available from the records taken at the time of firing, as noted in article 803 (d). The value of h_2 which is required for the determination of ΔX_2 can be found by applying a theorem of geometry, viz., that the length of the tangent to a circle from an exterior point is the mean proportional between the whole length of a secant from this point, and its external segment. In this case the length of the tangent is $O'P_1$, the whole length of the secant is $2r+h_2$ (r being the radius of the earth), and h_2 is the external segment. Since h_2 is insignificant in comparison with $2r$, we may use $2r$ as the value of the whole secant; also, no material error will result from using the uncorrected observed range X_0 in place of $O'P_1$. Hence we have

$$\frac{h_2}{X_0} = \frac{X_0}{2r}$$

and

$$h_2 = \frac{X_0^2}{2r} = [2.85590 - 10]X_0^2$$

the number in brackets being the colog of the value of $2r$ in yards. The formulas for the corrections to be applied to the observed range for height of gun and curvature of the earth therefore are, respectively,

$$\Delta X_1 = h_1 \cot \omega_0 \quad (803)$$

$$\Delta X_2 = h_2 \cot \omega_0 = [2.85590 - 10]X_0^2 \cot \omega_0. \quad (804)$$

Corrections for
height of gun
and curvature
of the earth

811. The following example illustrates the method of computing the corrections for height of gun and curvature of the earth, and gives, also, an idea of the usual magnitude of these corrections.

Given: In the experimental ranging of a certain gun at the proving ground, the average uncorrected observed range of five shots fired at angle of departure 15° was 22,842 yards, and their average measured initial velocity was 2608 f.s. For the firing at angle of departure 40° , the average uncorrected observed range for five shots was 38,088 yards and the average initial velocity was 2597 f.s. The height of the gun above tide level at the time of firing was 24 feet in both cases.

Find: The corrections to the observed range due to height of gun and curvature of the earth for each of the given groups, and the corrected observed ranges (assuming a wind correction of zero yards in each case).

Example of
reduction of
observed
range

Converting the given ranges and velocities into metric units and then entering Table VI with them, we can find the value of $\text{Log } C$ in each case and then use the latter to find ω for each case; this gives the following results.

For $\phi = 15^\circ$

$$X_0 = 22,842 \text{ yds.} = 20,887 \text{ m.}$$

$$V_0 = 2608 \text{ f.s.} = 794.9 \text{ m.s.}$$

$$\text{Log } C = 1.1646$$

$$\omega = 21^\circ 45'$$

For $\phi = 40^\circ$

$$X_0 = 38,088 \text{ yds.} = 34,828 \text{ m.}$$

$$V_0 = 2597 \text{ f.s.} = 791.6 \text{ m.s.}$$

$$\text{Log } C = 1.11590$$

$$\omega_0 = 52^\circ 00'$$

The value $h_1 = 24$ feet = 8 yards applies to both angles of departure. Hence we have all values required for the solution of (803) and (804), as follows.

For $\phi = 15^\circ$		For $\phi = 40^\circ$	
$h_1 = 8$ yds.....	$\log 0.90309$	$h_1 = 8$ yds.....	$\log 0.90309$
$\omega_0 = 21^\circ 45'$	$\text{lcot } 0.39907$	$\omega_0 = 52^\circ 00'$	$\text{lcot } 9.89281 - 10$
$\Delta X_1 = 20$ yds.....	$\log 1.30216$	$\Delta X_1 = 6$ yds.....	$\log 0.79590$
$X_0 = 22,842$ yds....	$2 \log 8.71748$	$X_0 = 38,088$ yds....	$2 \log 9.16158$
$\omega_0 = 21^\circ 45'$	$\text{lcot } 0.39907$	$\omega_0 = 52^\circ 00'$	$\text{lcot } 9.89281 - 10$
$2r =$	$\text{colog } 2.85590 - 10$	$2r =$	$\text{colog } 2.85590 - 10$
$\Delta X_2 = 94$ yds.....	$\log 1.97245$	$\Delta X_2 = 81$ yds.....	$\log 1.91029$
$X = 22,842 - 20 - 94 = 22,728$ yds.		$X = 38,088 - 6 - 81 = 38,001$ yds.	

812. It will be noted that the height of gun correction is greatest at short ranges, and becomes very small at long ranges due to the decrease in $\cot \omega$. In the correction for curvature of the earth, the increases of h_2 and decreases of $\cot \omega$, as the range increases, partially offset each other, and this correction remains within fairly narrow limits for practically all trajectories (for a given gun and initial velocity) up to about 40° , above which it decreases rapidly. It is of interest to note also the degree of error that results from the assumptions made in the foregoing article, viz., that for determining ΔX_1 the ω for point P_1 is the same as the ω at P_2 ; and that for determining ΔX_2 the uncorrected range X_0 may be used in place of $O'P_1$ (see Fig. 14). A second approximation, based on the determination of P_1 which is available from the first approximation made above, alters the final result only 0.6 yard for the 15° group and 0.3 yard for the 40° group. It is well to bear in mind that such devices as these often result in material simplification of a problem with but inconsequential sacrifice of accuracy. Thus the uncorrected observed range may be used also for finding the maximum ordinate which is required for determining the ballistic density and ballistic wind pertaining to the experimental ranging, and for determining the corrections to the observed range and lateral deviation due to wind, without causing material inaccuracy.

813. A question may be raised at this point as to the necessity for applying a correction for curvature of the earth to the observed range at the experimental ranging. It appears on first thought that the application of a correction for curvature of the earth at the experimental ranging serves merely to make it necessary to apply a similar correction in the subsequent service use of the gun, and that the latter might be avoided by omitting this correction in the experimental ranging in the first place. This would be true if the gun were to be laid in service firing by the same method that is employed at the proving ground, but this is not the case. In firing from a ship, the gun is laid in elevation with respect to the line of sight, and not by quadrant as at the proving ground. The effect of this difference in the manner of laying the gun will appear from a study of Figure 15.

In Figure 15 the gun is at O , at a height h above the water (as on the deck of a ship). OH is the horizontal plane through the gun, and $O'E$ is the surface of the water, on which a target is located at T . OT is the line of sight, and the gun sights are set so as to cause the gun to be elevated at the angle of elevation ϕ' which determines the trajectory that will intersect the line of sight at T , i.e., that will give the range OT measured in the line of sight. According to the as-

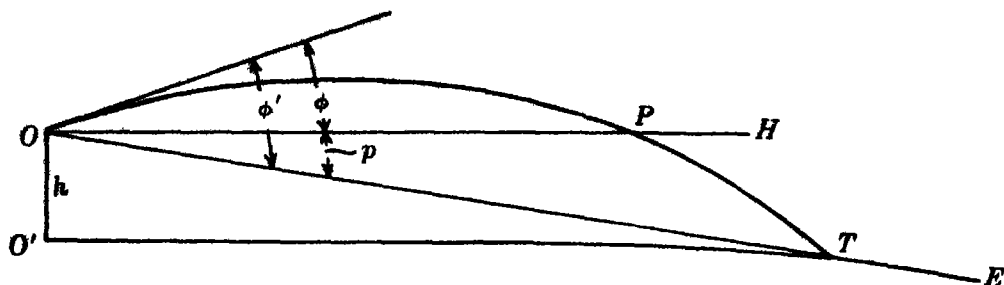


FIGURE 15

sumption of rigidity of the trajectory, as defined in article 319, it now appears that the situation here is the equivalent of having the gun and target in the same horizontal plane, and of using as the angle of elevation ϕ' the angle of departure ϕ tabulated in the range table for the given distance OT . This being the case, it is evident that neither the height of the gun nor the curvature of the earth enters into the situation, except insofar as they introduce a small angle of position. As shown in the discussion of the assumption of rigidity (arts. 317-319), the effect of such small angles of position as occur in practice is insignificant.* The assumed range-table condition of a horizontal plane containing both the gun and target is therefore directly applicable to surface fire when the gun is aimed at the target by means of sights.

A different situation obtains when indirect fire is used, for in this case gun elevation is laid with respect to the horizontal plane, as established by an instrument such as the stable-zenith director. Corrections for height of gun and curvature of the earth therefore are applicable in indirect fire, but the conditions under which indirect fire is used in naval practice ordinarily are such as to render corrections of this character of small importance in comparison with other problems that enter into this type of fire.

814. Having examined the methods by means of which the results of actual firings are observed at the experimental ranging conducted at the proving ground, we may now proceed to examine the problem of using this information for the construction of a range table. First of all we must find the value of i from the results of the experimental ranging, which is done as illustrated in the following problem.

Example of
experimental
determination
of i

Given: In the experimental ranging of the 16''2600 f.s. gun the following average results were obtained for a group of five shots fired at angle of departure 25° : uncorrected observed range, 31,845 yards; observed initial velocity, 2595 f.s.; observed weight of projectiles, 2099 lbs.; observed ballistic density, .961. The corrections to be applied to the observed range were found to be: for ballistic wind (—) 126 yards; for height of gun, (—) 10 yards; for curvature of the earth (—) 95 yards.

Find: The coefficient of form that corresponds to the above data.

* Since land artillery is usually laid in elevation by quadrant or the equivalent thereof, the U. S. Army range tables are based on the curved surface of the earth rather than on a horizontal plane. This is accomplished by allowing the effect of the earth's curvature to be subtracted from the ranges observed at the experimental ranging for range-table data.

The corrected observed range is

$$X = 31,845 - 126 - 10 - 95 = 31,614 \text{ yards.}$$

The corrected observed range and the observed initial velocity, converted to metric units, are

$$\begin{array}{ll} X = 31,614 \text{ yds.} \dots \log 4.49988 & V = 2595 \text{ f.s.} \dots \log 3.41414 \\ (\text{art. 705}) \dots \log 9.96114 - 10 & \log 9.48402 - 10 \\ \hline X = 28,908 \text{ m.} \dots \log 4.46102 & \\ \hline V = 790.97 \text{ m.s.} \dots \log 2.89816 & \end{array}$$

Using V to the nearest tenth of a meter-second, we now enter Table VI at the page headed 25° Range, and with the arguments $V = 791.0$ and $X = 28,908$ find the corresponding value of $\text{Log } C$ as follows

$$\begin{array}{l} \text{For } V = 791 \text{ and } \text{Log } C = 1.135 \\ \hline X = 28.607 + .1 \times 548 = 28,662 \end{array}$$

$$\begin{array}{l} \text{For } V = 791 \text{ and } \text{Log } C = 1.155 \\ \hline X = 29.143 + .1 \times 565 = 29,199 \end{array}$$

$$\text{For } V = 791 \text{ and } X = 28,908$$

$$\text{Log } C = 1.135 + \frac{246}{537} \times .020 = \underline{1.14416}.$$

From this value of $\text{Log } C$ we now deduce i by means of formula (406) transposed to the form

$$i = \frac{w}{\delta C d^2} \quad (805)$$

$$\begin{array}{ll} w = 2099 \dots \log 3.32201 & \\ \delta = .961 \dots \log 9.98272 - 10 & \text{colog } 0.01728 \\ C = \dots \log 1.14416 \dots & \text{colog } 8.85584 - 10 \\ d^2 = 256 \dots \log 2.40824 & \text{colog } 7.59176 - 10 \\ \hline i = .61220 \dots & \log 9.78689 - 10 \end{array}$$

815. The problem of determining the range-table values for the angle of departure 25° is simply the reverse of the above, using the i just determined, and the standard range-table values of V , w , and δ in place of the observed values that applied to the experimental ranging. The procedure is as illustrated in the following example.

Example of computation of range-table values

Given: The 16''2600 f.s. gun, for which the standard weight of projectile is 2100 lbs.; the coefficient of form, as determined by experimental ranging at the angle of departure 25° , is .61223.

Find: The range-table values of range (Col. 1), angle of fall (Col. 3), time of flight (Col. 4), striking velocity (Col. 5), and maximum ordinate (Col. 8), for the above gun at angle of departure 25° (Col. 2).

$$C = \frac{w}{\delta d^2} \quad (406)$$

$w=2100$	log 3.32222
$\delta=1.00$	log 0.00000.....colog 0.00000
$i=.61223$	log 9.78691-10.....colog 0.21309
$d^2=256$	log 2.40824.....colog 7.59176-10
$C=$	log 1.12707
$V=2600$ f.s.....	log 3.41497
(art. 705).....	log 9.48402-10
$V=792.49$ m.s.....	log 2.89899

The arguments for Table VI and Table VII are therefore $\phi=25^\circ$, $V=792.5$ m.s., and $\log C=1.12707$, and all that remains is to get the required values from these tables with the given arguments and to convert them to English units where necessary.

From Table VI we have

From the table headed $\phi=25^\circ$ Range

Range (Column 1)	For $V=790$, $X=28,069+.604 \times 538=28,394$
	For $V=800$, $X=28,601+.604 \times 554=28,936$
	For $V=792.5$, $X=28,394+\frac{1}{4} \times 542=28,530$ m.

From the table headed $\phi=25^\circ$ Time of Flight

Time of flight (Column 4)	For $V=790$, $T=5758+.604 \times 38=5781$
	For $V=800$, $T=5819+.604 \times 38=5842$
	For $V=792.5$, $T=5781+\frac{1}{4} \times 61=5796$ or <u>57.96 sec.</u>

From the table headed $\phi=25^\circ$ Angle of Fall

Angle of fall (Column 3)	For $V=790$, $\omega=4379-.604 \times 44=4352$
	For $V=800$, $\omega=4396-.604 \times 45=4369$
	For $V=792.5$, $\omega=4352+\frac{1}{4} \times 17=4356$ or <u>$36^\circ 18'$</u>

From the table headed $\phi=25^\circ$ Terminal Velocity

Striking velocity (Column 5)	For $V=790$, $v_w=4223+.604 \times 73=4267$
	For $V=800$, $v_w=4244+.604 \times 75=4289$
	For $V=792.5$, $v_w=4267+\frac{1}{4} \times 22=4273$ or <u>427.3 m.s.</u>

From Table VII, using the pages headed $V=760$ m.s. and $V=800$ m.s. we have

Maximum ordinate (Column 8)	For $V=760$, $y_s=3815+.541 \times 111=3875$
	For $V=800$, $y_s=4153+.541 \times 127=4222$
	For $V=792.5$, $y_s=3875+.812 \times 347=4157$ m.

Converting to English units, we have

$X=28,530$ m.....	log 4.45530
$y_s=4157$ m.....	log 3.61878
$v_w=427.3$ m.s.....	log 2.63073
(art. 705).....	log 0.03886 log 0.51598 log 0.51598
$X=31,201$ yds.....	log 4.49416
$y_s=13,638$ ft.....	log 4.13476
$v_w=1401.9$ f.s.....	log 3.14671

These results may now be tabulated as follows, in units and number of places as in the U. S. Navy range tables.

Range	Angle of departure	Angle of fall	Time of flight	Striking velocity	Maximum ordinate
1	2	3	4	5	6
yards	° ' ''	° ' ''	seconds	f.s.	feet
31,201	25 00.0	36 18	57.96	1402	13,638

816. Very close agreement will be noted between the above results and those tabulated in the 16''2600 f.s. range table (see *Range and Ballistic Tables, 1935*), except in the case of the maximum ordinate, for which the latter gives 13,555 feet as compared to 13,638 feet as found above. This difference (83 feet) is due principally to the fact that the untranslated French edition of the A.L.V.F. Tables (1921) was used for calculating the maximum ordinate for this range table, whereas we have used the War Department Tables for this element (see arts. 702, 703, and 714). As will be noted from inspection of Columns 1 and 6 of the range table, a difference of 83 feet in the maximum ordinate corresponds to a difference of about 60 yards in the range, or about $\frac{1}{5}\%$ of the range.* The other very slight differences between the above results and the range-table values are due to smoothing operations that enter into the final tabulation of the latter, which will be described presently.

817. In the same manner as illustrated above for the angle of departure 25°, values of i are determined for each of the angles of departure at which experi-

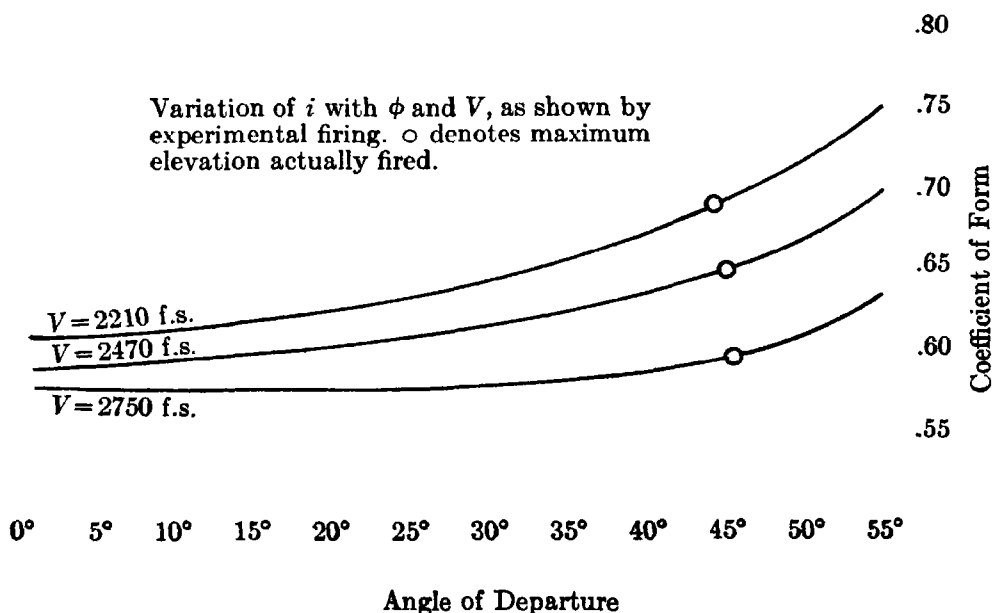


FIGURE 16

* It should be understood, however, that this difference results from the use of the same value of i for two different tables. Exact agreement with the value $y_m = 13,555$ feet given by the A.L.V.F. Tables (1921) can be secured by using $i = .6261$ in connection with Table VII in this case. On the whole, the agreement between results obtained from the two tables with the same value of i is remarkably good, considering the differences that exist between the retardation functions, altitude-density functions, values of gravity, and computational methods, on which the two systems of tables are based (ref. arts. 623, 702, 714).

mental ranging is done. A good estimate of the values of i pertaining to angles of departure not included in the experimental ranging may then be found by interpolating between the experimentally determined values; for this purpose a graph of i plotted against ϕ , faired through the several points established by the experimental ranging, is most convenient.

For reasons outlined in article 417, it is evident that the value of i may be expected to vary with both V and ϕ . Figure 16 illustrates the magnitude of such variations, as determined experimentally for a certain 16" gun at full charge and at two reduced charges, the design and weight of projectile remaining the same. In this particular case the variation in i is inappreciable up to about $\phi = 35^\circ$ for the full-charge initial velocity, while for the reduced initial velocities it becomes noticeable above about $\phi = 10^\circ$. This may be attributed to the fact that the combination of projectile and rifling design produce a more uniform flight stability for the maximum initial velocity than for the others. In any event, the experimental ranging reveals the conditions as they actually exist, and the necessary values of i for various angles of departure may be found from graphs, such as illustrated in Figure 16, drawn to large scale.

818. It is to be noted that the variation of i with initial velocity is great enough to require separate experimental rangings for changes of initial velocity that result from the use of different charges (such as the regular service charge and the reduced target-practice charges). It should be apparent, also, that separate experimental rangings and separate range tables may be required for different projectiles fired from the same gun, even though the initial velocities remain the same. In some cases the differences in projectiles do not affect their flight performance, and hence separate range tables are not required. For example, target-practice projectiles are designed to conform closely to the flight performance of the service projectiles, and the same range table is applicable to both. In other cases the difference between projectiles is very great, as in the case of common and flat-nose projectiles, and the range table for one type of projectile is then practically worthless for another type.

819. Having determined values of i for all angles of departure that are to be included in the range table, the process of finding the values of range, angle of fall, time of flight, striking velocity, and maximum ordinate for any angle of departure is exactly as illustrated in article 815.* Range-table values are computed in this manner for 5°-increments in angle of departure (i.e., for the tabular values of ϕ given in the A.L.V.F. Tables). The computed values for the several elements are then plotted against angle of departure, to large scale, and a graph is faired through the computed points for each element. From the graph of range against angle of departure, the angle of departure is found for each tabular value of range (in increments of 100 yards). The values picked from the graph are adjusted in order to make the second differences run smoothly; this serves to check the accuracy with which the intermediate points were picked from the graph. A similar process is followed for the other elements, the latter being tabulated, eventually, against range as argument. The general character of these graphs is illustrated in Plate IV.

The process of fairing curves through the computed points, and the subse-

* See articles 513-514, and foot-note appearing under article 714. The work for angles of departure below 15° is essentially the same as here outlined except, of course, that the computed points are based on Siacci's Method. In view of the prospective adoption, in the near future, of numerical integration ballistic tables for all angles of departure, it is considered desirable to confine further references to the applications of Siacci's Method, at this time, to an Appendix.

quent process of smoothing out the second differences of values picked from the curves, naturally give rise to slight differences between results that eventually are tabulated in the range table and those which are found by computation for a specific point. This explains the slight differences which have been noted in article 816.

820. Although consideration of the drift of the projectile which results from the spin imparted to it by the rifling of the gun is not to be taken up at present, it is appropriate to mention at this point that the computation of the drift depends eventually upon experimental determination of a drift coefficient (D') which, in character and purpose, is not unlike the coefficient of form i .

Determination of observed drift The drift coefficient must be given such a value that, when substituted in the drift formula (901), it causes agreement between the computed drift and the observed drift. The observed drift is determined at the experimental ranging by correcting the observed lateral deviation (see art. 803 (d)) for the effect of wind across the line of fire. The problem of finding the drift coefficient from the observed drift will be taken up in Chapter 9.

EXERCISES*

- Given:* In the experimental ranging of a 16'' gun for the determination of range-table data for the service velocity of 2600 f.s., the following observed values were obtained from a group of five shots fired at angle of departure 35°: observed initial velocity, 2592 f.s.; uncorrected observed range, 36,026 yards; observed weight of projectiles, 2103 lbs.; observed ballistic density 1.035. The corrections to be applied to the observed range were found to be as follows: for ballistic wind, (+) 151 yards; for height of gun, (−) 7 yards; for curvature of the earth, (−) 89 yards.

Find: (a) The coefficient of form that corresponds to the above data. (b) The range-table range for this gun at angle of departure 35°.

Answers: (a) $i = .61296$

(b) $X = 36,801$ yards.

- Given:* In the experimental ranging of a 16'' gun for the determination of range-table data for the target-practice velocity of 2000 f.s., the following observed values were obtained from a group of five shots fired at angle of departure 35°: observed initial velocity, 1985 f.s.; uncorrected observed range, 23,779 yards; observed weight of projectiles, 2096 lbs.; observed ballistic density, .984. The corrections to be applied to the observed range were found to be as follows: for ballistic wind (+) 127 yards; for height of gun (−) 9 yards; for curvature of the earth, (−) 42 yards.

Find: (a) The coefficient of form that corresponds to the above data. (b) The range-table range for this gun at the angle of departure 35°.

Answers: (a) $i = .67920$

(b) $X = 23,994$ yards.

- Given:* The initial velocity, diameter, weight, and coefficient of form of the projectile, and the angle of departure.

Find: The values for the following range-table columns for the given angles of departure: Column 1 (Range); Column 3 (Angle of fall); Column 4 (Time of flight); Column 5 (Striking velocity); Column 8 (Maximum ordinate).

* Use V to the nearest tenth of a meter-second in all cases.

	Given					Answers				
	V f.s.	d in.	w lbs.	i	ϕ	X yds.	ω	T sec.	v_w f.s.	y^* ft.
A	2600	16	2100	.61230	15°	22,926	21° 34'	37.10	1485	5571
B	2600	16	2100	.61200	30	34,328	42 19	67.65	1423	18,589
C	2600	16	2100	.61140	40	38,630	51 48	85.76	1504	29,842

* See art. 816.

4. From the 16''2600 f.s. range table given in *Range and Ballistics Tables, 1935*, deduce the values of i that correspond to the data of this range table at the following angles of departure: (a) 20°; (b) 35°.

Answers: (a) $i = .61244$

(b) $i = .61280$

CHAPTER 9

THE DEVIATION OF THE TRAJECTORY FROM A PLANE CURVE; THE DRIFT. THE DANGER SPACE. CHANGE IN HEIGHT OF IMPACT. THE DETERMINATION OF THE QUANTITIES IN COLUMNS 6, 7, AND 19 OF THE RANGE TABLE.

New Symbols Introduced

D . . . The drift of the projectile.

D' . . . The drift coefficient used in determining the drift.

μ . . . The final twist of the rifling of the gun, i.e., the length of bore in calibers corresponding to one complete turn of the rifling at its final twist.

S . . . The danger space (art. 914).

901. Having proceeded in the foregoing discussions under the assumption that the axis of the projectile remains coincident with the tangent to the trajectory (art. 213 (5)), we have developed solutions for the trajectory considered as a plane curve confined to the vertical plane containing the line of departure. We know, however, that the trajectory actually is not a plane curve, and that it deviates from the vertical plane through the line of departure by ever increasing

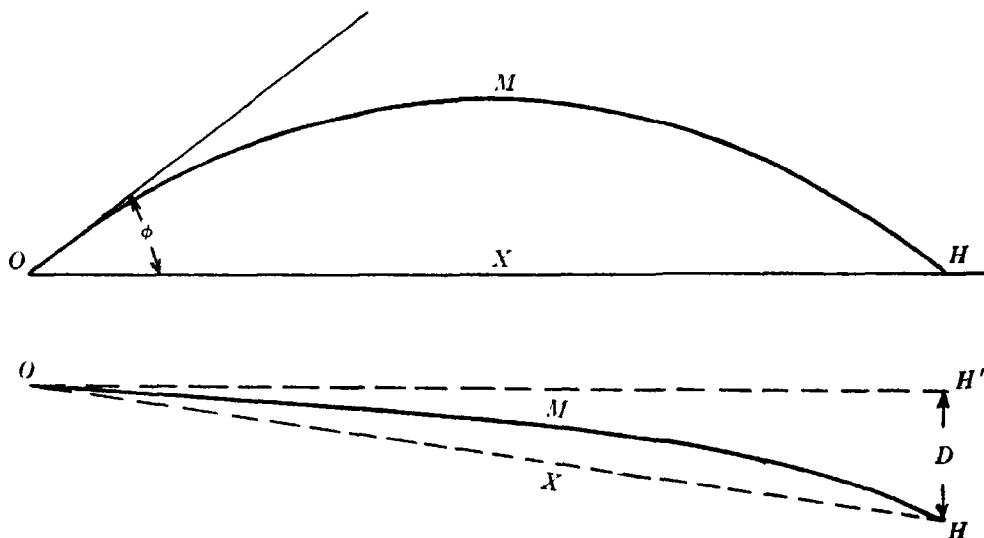


FIGURE 17

amounts as the point of fall is approached. In other words, the tangent to the trajectory constantly changes its direction not only with respect to the horizontal plane, but also with respect to the original plane of fire. This is illustrated in Figure 17, in which the upper diagram shows the side elevation and the lower diagram the plan view of the same trajectory. In the plan view, OH' represents the trace of the original plane of fire, and OMH the trace of the trajectory. The amount of the deviation of the curve OMH from the straight line OH' depends both on the lateral component of any wind that may be acting on the projectile, and on lateral

forces which result from its rotation. The amount of this deviation at the point of fall, under the standard condition of no wind, is called the *drift* and denoted by the symbol D (ordinarily expressed in yards). *The drift, then, is the lateral deviation*

of the point of fall from the original plane of fire, due only to effects set up by the rotation of the projectile. Other lateral deviations, such as that due to wind, should not be termed as being part of the drift.

The drift is measured as illustrated in Figure 17, where D is the perpendicular distance of the actual point of fall H from the original plane of fire OH' .

902. A study of Figure 17 will show that the point of fall H' , as predicted by a computation based on the assumption of a trajectory remaining within the original plane of fire, is located at a considerable lateral distance from the actual point of fall H . The lateral error D , however, although itself of considerable magnitude, may nevertheless cause only a slight error in the computed range. In other words,

the difference between the computed range OH' and the actual range OH may be slight even if the distance $H'H = D$ is considerable. The following examples bear this out. The 16'' 2600 f.s. range table shows that for this gun the drift is 38.3 yards at the range 10,000 yards; in this case the difference between OH' and OH (Fig. 17) is insignificant (actually only .08 yard). For trajectories limited to angles of departure not exceeding 45° , the drift does not exceed about 5% of the range, in which case the difference between OH' and OH is limited to about 0.1%.

It should be clear, then, that no appreciable error has been occasioned by our assumption of a plane trajectory up to this point, insofar as determination of the range is concerned; and it is to be presumed that a similar situation exists with respect to the other elements which have been dealt with under the same assumption. It is to be noted, however, that the difference between computed and actual results, as dealt with above, is in fact only a theoretical difference, since the value of i that is used for obtaining the computed range is based on an observed range referred to the actual point of fall (art. 803 (d)). Whatever distinction theoretically may exist between OH' and OH of Figure 17 therefore is fully accounted for by i . The same is true of any range effects which ensue from the rotation of the projectile, due to vertical forces set up by the projectile's obliquity to its direction of flight; this has already been discussed in article 417.

903. In proceeding to a study of the causes of the drift of a projectile, it may be remarked, first of all, that the complete explanation of this phenomenon rests on conjecture probably to a greater degree than is the case with any other phase of ballistics. One finds, in connection with some details of this explanation, a very considerable divergence of opinion among eminent authorities.* The explanations offered here are but very general in character, and are based on a broad view of the situation. What may be a satisfactory explanation of the behavior of a rotating projectile in one specific instance, may fail to apply in another. The expository treatment given herein gives consideration to a number of hypotheses under which the causes of the drift have been investigated with the aid of mathe-

* Exhaustive mathematical treatments of this subject are given on pp. 308-360, *Handbook of Ballistics*, Vol. I, Crans and Becker; on pp. 172-257, *New Methods in Exterior Ballistics*, Moulton; in *The Aerodynamics of a Spinning Shell*, Fowler, Gallop, Lock, and Richmond (Philosophical Transactions of the Royal Society of London, Vol. 221, 1921); and in *The Damping Effect on a Rotating Projectile Due to the Path of the Center of Gravity*, Guion (Journal of the Maryland Academy of Sciences, Vol. I, 1930).

mathematical analysis. These analyses are much too lengthy for inclusion in this text, but may be found in the references noted on page 106.

The drift of an elongated, rotating projectile may be considered to result from three causes, viz:

- | | |
|------------------------|--|
| Causes of drift | (a) Gyroscopic action. |
| | (b) The action of air adhering to the projectile. |
| | (c) The cushioning action of air banking up on one side of the projectile. |

It is reasonably certain that the latter two causes have only a very minor effect as compared to the first, and we shall therefore dismiss them with a brief discussion of their nature.

904. The effect of air adhering to the rotating projectile is similar to that which is employed to propel a rotor ship; it is often referred to as the *Magnus Effect*.^{*} In the case of a projectile, the initial tendency of the projectile to maintain the original direction of its axis, while the tangent to the trajectory moves downward, causes the air stream to strike the projectile's under-side. With right-handed spin, the air adhering to the right-hand side of the projectile, moving at the peripheral speed of the latter, then opposes the air stream created by the projectile's flight, and the result is an increase of pressure on the right-hand side. Similarly, on the left-hand side there is a rarefaction, and the projectile tends to move to the side of lesser pressure, i.e., to the left. If the axis of the projectile should change its direction and point below the tangent, the effect would be reversed. As the axis does in fact change its position with respect to the tangent, as we shall see presently, the effect of the adhering air may result in deviations of the projectile first to one side and then to the other, as well as up and down, but the initial movement is to the left, for right-handed spin. The Magnus Effect, in addition to its well-known application to rotor ships, serves also to explain the curves of a pitched baseball, and the "hooks" and "slices" of a golf-ball. However, considering the small obliquity of a projectile with respect to the direction of its flight, only a small component of the velocity of the adhering air is presented to the air stream; this fact, plus the probable alternations in direction of the resulting movements of the projectile, point to the conclusion that the Magnus Effect has but a small part in accounting for the observed drift.

The cushioning effect	The theory of the so-called cushioning effect [†] also depends upon obliquity of the projectile. If the under-side is presented to the air stream, the air banks up against this side, forming a sort of cushion against which the spinning projectile rolls by virtue of the greater friction of this relatively dense air cushion as compared to the condition on the opposite side. What has been said with regard to alternations of direction of the Magnus Effect also applies here, but in the case of the cushioning effect the initial movement is to the right for right-handed spin; hence these two effects oppose each other. It is probable, however, that the latter effect is of even less importance in accounting for the drift than is the Magnus Effect.
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905. It is generally accepted that the principal cause of the drift lies in the gyroscopic properties of the rapidly spinning projectile. According to the laws of

^{*} From G. Magnus who, in about 1850-52, made important contributions to the theory of drift. Magnus not only investigated the effect which now bears his name, but also was among the first to investigate the gyroscopic theory.

[†] Sometimes referred to as the *Poisson Effect*, after S. D. Poisson who, in 1839, first offered it as a contributory cause of the drift.

**General
features of
gyroscopic
action**

the gyroscope, the projectile seeks, first of all, to maintain its axis in the direction of the line of departure. The center of gravity of the projectile, however, follows the curved path of the trajectory, and the instantaneous direction of its motion, at any point, is that of the tangent to the trajectory at that point. The projectile's tendency to maintain the original direction of its axis therefore soon results in leaving this axis pointed slightly above the tangent to the trajectory, and the force of the air resistance opposed to the flight of the projectile is then applied against the latter's underside. This is illustrated in Figure 18 (a), in which GT is the tangent to the trajectory at the point which is occupied by the center of gravity G of the projectile. If the center of pressure of the surface exposed to the action of the air resistance were so located as to lie always in line with the center of gravity G , the condition illustrated would continue, with the axis maintaining its original direction and making an ever increasing angle with the tangent.

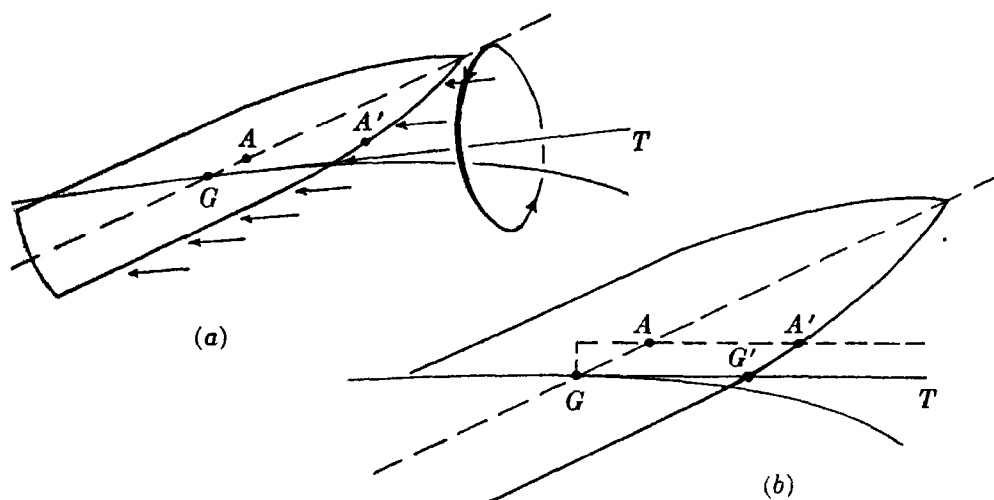


FIGURE 18

However, the relation between the weight distribution and surface of elongated, ogival-headed projectiles is such as to bring about the condition illustrated in Figure 18 (b). Here A' is the center of pressure of the surface that is being acted upon by the force of air resistance, and $A'A$ is the line along which this force acts. $A'A$ meets the axis of the projectile at A , which lies between the center of gravity and tip of the projectile. An overturning moment is thus set up, tending to tumble the projectile end over end, but the gyroscopic force of the projectile opposes this moment and tends to keep the axis in the original direction. According to the laws of the gyroscope, the action of the projectile in seeking to overcome this overturning moment must manifest itself in a precession of the projectile about the direction of the force which creates the moment; also according to these laws, the precessional revolution, for a projectile having right-handed spin, must be clockwise as viewed from the rear (see Figure 18 (a)).

The general features of gyroscopic action, as outlined above, can be demonstrated readily with any gyroscopic device, as for example a top. In 1911, the Bureau of Ordnance, U. S. Navy Department, made a small model projectile and suspended it in gimbals. The model was then rotated at a speed comparable to the speed of rotation of a projectile. To simulate the air resistance in flight, a blast of air was directed against the rotating model, and the latter's behavior confirmed the above

**Laboratory
reproductions
of gyroscopic
action**

theory.* As soon as the blast of air met the rotating model, the latter began to precess about the direction of the air. As the direction of the blast was changed to represent the changing direction of the air resistance in flight, the model's axis depressed to meet the new direction of the air blast.

906. Under the condition illustrated in Figure 18(a), i.e., with the projectile's axis already at a considerable angle with respect to the tangent, and the latter changing its direction downward at a slow rate, the projectile would undoubtedly make complete precessional revolutions about the tangent, as illustrated. However, the actual situation is not as simple as this. Assuming that the projection from the gun is perfectly regular, so that the projectile commences its flight exactly head-on, no precession at all takes place until the tangent drops below the projectile's axis. As soon as this occurs, the projectile's axis starts to move in the right-hand semicircle of its precessional revolution, and the axis therefore soon is moving downward, i.e., in the same direction as the tangent. The rate at which the axis moves, i.e., its precessional period, depends chiefly upon the projectile's physical features and rate of spin (it varies inversely as the latter). It is readily conceivable, therefore, that the character of the projectile's precessional motion with respect to the tangent depends materially upon the relation that exists between the rates at which the axis and tangent are changing direction.

A general conception of this situation may be gained by studying the motion of the axis through a complete cycle. The tangent having dropped sufficiently below the axis to set up precession, the axis moves first to the right and then down. In the early part of the first quadrant of this revolution, the downward component of motion of the axis is less than that of the tangent, but as the axis turns farther its downward component increases, and eventually the axis overtakes and passes the tangent. As the end of the second quadrant is approached, the downward component of the axis eventually decreases again to the point where it becomes less than that of the tangent, and the latter then tends to overtake the axis.† Assuming, however, that the tangent does not fully overtake the axis at the end of the second quadrant, the latter then enters the third quadrant, moving to the left of the tangent and then up, and eventually completes the whole revolution. But in the entire left-hand semicircle of this revolution, the motion of the axis is contrary to that of the tangent. Hence this semicircle is of shorter duration than the right-hand semicircle, and it follows that the axis of the projectile must, in any event, be a longer time to the right of the tangent than to the left.

907. The very general analysis given above serves to prove that even if the projectile actually makes complete revolutions about the tangent, and deviates alternately to the right and left as the result of the pressure exerted against its sides, there is nevertheless a preponderance of such deviations toward the side on which the axis and tangent are both moving downward, and for right-handed spin this is to the right (with respect to the direction of flight). In short, it is the constant dipping of the tangent, in combination with the precessional motion of the axis, that accounts

The character
of the motion
of the axis
with respect
to the tangent

Why the drift
preponderates
to one side

* Ref. p. 36, *Ordnance Pamphlet No. 399* (U. S. Navy, 1912).

† Dr. Cranz, in his *Handbook of Ballistics*, Vol. I, (pp. 328-329), concludes that, "Neglecting nutation, the point of the shell describes in space a cycloidal curve, and the axis of the shell a cycloidal cone, lying on the right-hand side of the vertical plane through the tangent"; also, "And the fact that the point of the shell, after completing a cycloidal arc, always comes into coincidence with the tangent to the path, is the reason why the shell strikes the ground with its point." On pp. 345-350, he offers a graphical step-by-step solution of the motion of the axis with respect to the tangent, by means of which he supports these conclusions.

both for the drift and for the fact that the axis of a properly designed projectile never deviates very far from the tangent.

The fact that the axis and tangent are approaching coincidence in the second quadrant of the precessional revolution lends support to the theory that the motion of the axis, and hence the sidewise deviation of the projectile, are cycloidal in nature (see footnote, p. 109). According to this theory, the axis actually comes into coincidence with the tangent at the end of each half revolution, and there is no deviation at all to the left. In the absence of rigorous proof, it seems debatable whether this condition can be accepted as universally applicable, although it is readily conceivable that it may be approached to varying degrees, according to the immediate relation between the stability of the projectile and the curvature of the trajectory.

A conclusion arrived at by some eminent authorities, as the result of careful mathematical analysis, is that, as a consequence of the dipping of the tangent, the precessional motion tends to damp itself out and to approach a condition in which the projectile's axis, somewhere in the right-hand semi-circle of its precessional revolution, practically keeps pace with the downward motion of the tangent.* Under this condition the axis then remains pointed slightly to the right of the tangent, and the projectile yaws continuously to the right. The angle at which the axis tends to steady itself, i.e., the angle of yaw, evidently depends upon the rate at which the tangent is moving downward, and hence must increase as the curvature of the trajectory increases. This is in accord with the observed fact that the drift increases not at a constant rate with respect to the range, but at an ever increasing rate. (It increases, in fact, approximately in proportion to the square of the time of flight.) The observed nature of the drift, in this respect, is accounted for also under the conception of cycloidal oscillations, or of complete precessional revolutions, of the projectile's axis about the tangent. For in the former case the duration of each yaw to the right, and in the latter the preponderance of the yaws to the right over those to the left, likewise depend on the rate at which the tangent is moving downward.

A close analysis of the behavior of a rapidly spinning projectile in flight is further complicated by the frictional effects already mentioned in article 904, and by effects resulting from irregular projection. The latter may have a very marked influence on the motion of the projectile, especially in the early stages of its flight. The projectile may emerge from the bore with its axis already slightly canted with respect to the tangent, or it may, while still in the region of the muzzle blast, be thrown aslant by the column of gases acting against its base. Precession results also from any such initial inclination of the projectile's axis, but obviously it does not progress in the same relation with respect to the motion of the tangent as does the precession which results directly from the latter. Also, the precessional motion that is set up by irregularities in the initial stages of flight eventually is damped out by the regular precession, which alone is sustained by a continuing cause.

908. The gyroscopic stability of a projectile can be controlled within fairly

* In *New Methods in Exterior Ballistics* (p. 254), Dr. Moulton concludes that, "The oscillations of the projectile are damped toward the solution given in . . . , and in this solution, for guns having right-hand rifling, the axis of the projectile is to the right of the plane of fire and a little below the tangent to the trajectory." A similar result is stated in *The Aerodynamics of a Spinning Shell*, by Fowler, Gallop, Lock, and Richmond.

General considerations governing the degree of stability desired in a projectile

wide limits by means of the initial spin given to the projectile by the gun, i.e., by means of the twist of the rifling. Other things remaining the same, a greater rate of spin causes greater stability. It is not to be supposed, however, that great stability (in the sense here implied) is a desirable feature, for with infinitely great stability the projectile would resist precession altogether and consequently would fail to fly head-on. With relatively very great stability, precession would be very slow, the amplitude of the precessional arc relatively great, and the lateral deviation excessive and perhaps erratic; moreover, under this condition the irregularities of flight incident to canted ejection would persist for a longer time and consequently produce greater errors in the point of fall. What is to be desired, rather, is a degree of stability sufficiently great to resist the overturning moment (Figure 18(b)), but not so great as to prevent precession of a small amplitude about the tangent. The attainment of a satisfactory balance in these respects rests chiefly on proving-ground experiments designed to afford immediate information as to the flight characteristics of projectiles. These experiments involve the firing of projectiles through a series of cardboard screens, and careful measurement of the holes left by them in the several screens, whence the obliquity of the projectiles at these several points in their flight may be deduced.* Regularity in the points of fall at all ranges is, of course, the ultimate criterion as to the acceptability of a given design.

909. The attainment of an ideal degree of stability for a projectile is complicated by the fact that the conditions which define this ideal do not remain constant throughout the trajectory. As the remaining velocity of the projectile decreases, the overturning moment (Figure 18 (b)) and hence the degree of stability required to oppose this moment, also decrease. If the relative stability of the projectile is to remain uniform throughout the trajectory, it is necessary, therefore, that the rate of spin decrease in the same proportion as the remaining velocity decreases. However, this is not the case, since the rate of spin decreases less rapidly than does the remaining velocity, and consequently the relative stability of the projectile increases as the latter proceeds in its flight. Moreover, as the projectile ascends to air of lesser density the overturning moment decreases, and the relative stability increases in the ascending branch of the trajectory due also to this cause. This is of special significance in the case of trajectories having a large angle of departure, since in this case the increased stability of the projectile in the vicinity of the maximum ordinate is particularly undesirable in view of the relatively great rate of change of curvature in that region. The design of a projectile, with respect to its stability characteristics, therefore must be a compromise between the requirements for short trajectories and long trajectories.†

Apart from the lateral effects which result from the oscillations of the projectile's axis about the tangent to the trajectory and which manifest themselves in the drift, vertical effects evidently also result therefrom. That is, the varying angle of inclination of the axis with respect to the tangent causes deviations in the

* A description of such experiments is given in *Ordnance Pamphlet No. 399* (U. S. Navy, 1912), and in *The Aerodynamics of a Spinning Shell*, Fowler, Gallop, Lock, and Richmond.

† Experiments have recently been made with projectiles whose ogives are fitted with wind vanes, with a view to creating greater uniformity in the stability of the projectile throughout the trajectory by causing its rate of spin to be retarded more nearly in proportion to the retardation of the velocity of translation.

vertical plane as well as in the horizontal plane, and the rotation of the projectile therefore affects the range. The angle of inclination between the axis and tangent also modifies the effective cross-sectional area of the projectile, which affects the retardation, and hence the range. Variations in the mean effective stability factor for various trajectories of the same gun and projectile, arising from the considerations outlined in the foregoing paragraph, thus contribute largely to the variations in coefficient of form with angle of departure that are noted from the results of experimental ranging. In short, the rotation of the projectile, although it is commonly associated only with the phenomenon of drift, has a very important bearing on the range as well.

910. Analytical expressions for the drift have been deduced under the hypothesis of gyroscopic action, but so little is known about the actual values of some of the physical factors involved in them, under the varying conditions that exist in a whole trajectory, that the practical solution for the drift eventually depends on expressions that are highly empirical in nature. As has already been observed above, the behavior of the projectile under the influences set up by its rotation accounts largely for the empirical nature of the coefficient of form, i , which enters into the equation to the trajectory as referred to the vertical plane. Considering, now, that this behavior of the projectile is but one of several factors that influence its path with respect to the vertical plane, whereas it is the sole cause of the lateral deviation from this plane (under the assumed condition of no wind), it can readily be appreciated that the entire system of solution for this lateral deviation is empirical to a much greater degree than is the case with the solutions for elements that have previously been dealt with. The relations indicated by an analytical treatment of the drift under the gyroscopic theory, are incorporated to varying degrees in the numerous expressions for this element that have been proposed from time to time. But it is characteristic of all such expressions that the analytical terms contained in them are usually rather broad approximations, based on averages pertaining to entire trajectories and sometimes even on averages pertaining generally to all projectiles of similar type. And, in practically all cases, the expressions depend also on purely empirical coefficients which must be determined from the results of experimental firing quite in the same manner as is the case with the coefficient of form, i .

911. The drift formula that has been used for the more recent of the range tables that appear in *Range and Ballistic Tables, 1935*, is given here as an example.* This formula is due to Colonel A. Hamilton, U. S. Army.

* A drift formula devised by Mr. E. B. Scott, of the U. S. Naval Proving Ground staff, has been used in connection with some of the later range tables. The general expression for this formula is

$$D = \frac{T^2}{a + \frac{b\mu}{V} + \frac{\phi}{c}}$$

in which a , b , and c are coefficients that are to be determined by experiment. The values $a = .871$, $b = 349.4$, and $c = 128.2$ have been found to give very satisfactory agreement with observed results, under a wide variety of conditions, with projectiles $3\frac{1}{2}$ calibers long (these values require ϕ to be expressed in degrees and give D in yards). For other projectiles different values of a , b , c , are required, but satisfactory results are obtained also by applying an additional coefficient (of the nature of D') to the entire expression, with values of a , b , c as given above. For additional formulas, see pp. 350-356, *Handbook of Ballistics*, Cranz and Becker; also pp. 118-121, *Computation of Firing Tables for the U. S. Army*, H. P. Hitchcock.

The very simple formula $D = kT^2$, in which k is a constant that is to be determined

Hamilton's
drift
formula

$$D = X(1 - D') \frac{d^3}{\mu w} (\phi^\circ + \omega^\circ) \sec \phi. \quad (901)$$

In the above formula ϕ , ω , d , w , and X have their usual meanings, but the values of ϕ and ω within the bracket are to be expressed in radians. D is the drift, in yards or in feet according to whether X is expressed in yards or feet. The factor μ represents the final twist of the rifling of the gun, expressed in terms of the length of bore in calibers covered by one complete turn of the rifling at its final twist.

The factor D' , which we shall call the *drift coefficient*, is purely empirical in nature. Its value is so chosen that when it is substituted in the drift formula (901),

it will cause agreement between the computed drift and the observed drift as determined by experimental ranging. It should be apparent, then, that the principles governing the determination of D' are quite similar to those which are involved in the determination of i . Also, that the value of D' thus determined partakes of the character of i , i.e., it accounts for any influences which have contributed to the observed drift but have not separately been accounted for in the drift formula. The effect of lateral jump, for example, is included in the value of D' . The value of D' , however, depends chiefly on the form of the projectile, and it may be thought of as a sort of coefficient of form for drift. Its value for U. S. Navy projectiles is usually around .7.

912. In the course of experimental ranging, the uncorrected observed lateral deviation is determined as already explained in article 803 (d); by applying to this a correction for the component of wind across the line of fire that existed at the time of the ranging, the observed drift is found. The latter is then substituted in (901) and the required value of D' is found. However, since we have no means for correcting this observed drift as would be necessary to refer it to the *corrected* observed range, we use in (901) the range at which the observed drift actually was measured, i.e., the *uncorrected* observed range, and for ϕ and ω we use the values corresponding to the latter as determined from subsequent computations of the range-table data. An example will illustrate these features.

Experimental determination of the drift coefficient *Given:* In the experimental ranging of the 16'' 2600 f.s. gun the following average results were obtained for a group of five shots fired at angle of departure 25° : uncorrected observed range, 31,845 yards; observed weight of projectiles 2099 lbs.; observed lateral deviation, 747 yards to the right; correction for observed ballistic wind component of 4 knots across the line of fire, 42 yards to the right. By subsequent computation the angle of departure for the range 31,845 yards was found to be $25^\circ 57'$, and the angle of fall $37^\circ 27'$. The final twist of rifling in the gun was one turn in 32 calibers.

Find: The drift coefficient D' .

First we shall transpose (901), and at the same time introduce the relation $57.3 = 1$ radian in order that the values of ϕ and ω in the bracket may be expressed in degrees. We have then

$$(1 - D') = \frac{D}{X} \times \frac{\mu w}{d^3} \left(\frac{57.3}{\phi + \omega} \right) \cos \phi \quad (902)$$

experimentally and D and T have their usual meanings, has also been found to give very satisfactory results in connection with long projectiles, and has been used in computing Column 6 of some tables.

$$D = 747 + 42 = 789 \text{ yards}$$

$$\phi + \omega = 63^\circ 24' = 63^\circ 40'$$

$D = 789$	log 2.89708
$X = 31,845$	log 4.50304
$\mu = 32$	colog 5.49696 - 10
$w = 2099$	log 1.50515
$d^2 = 4096$	log 3.32201
57.3	colog 6.38764 - 10
$(\phi + \omega) = 63^\circ 40'$	log 1.75815
$\phi = 25^\circ 57'$	log 1.80209
$1 - D' = .33017$	colog 8.19791 - 10
$D' = .66983$	lcos 9.95384 - 10
		log 9.51874 - 10

913. The computation of the drift for the range table (Column 6) is simply the reverse of the above, using standard range-table values. The coefficient D' is found to remain nearly enough constant to permit the use of an average value for a wide range of angles of departure; let us assume, therefore, that $D' = .67$ is the average determined from the experimental ranging, and use this to compute the drift for the angle of departure 30° , at which ranging shots were not fired. The problem is stated as follows.

Given: The 16'' 2600 f.s. gun, for which the standard weight of projectile is 2100 lbs., the drift coefficient is .67, and the final twist of rifling one turn in 32 calibers.

Find: The range-table value of the drift (Column 6) for this gun at the angle of departure 30° , for which the range-table value of the range is 34,329 yards, and of angle of fall $42^\circ 21'$.

For this solution we shall write (901) as follows

$$D = X(1 - D') \frac{d^2}{\mu w} \left(\frac{\phi + \omega}{57.3} \right) \sec \phi \quad (901A)$$

$$\phi + \omega = 72^\circ 21' = 72^\circ 35'$$

$X = 34,329$	log 4.53566
$(1 - D') = .33$	log 9.51851 - 10
$d^2 = 4096$	log 3.61236
$\mu = 32$	log 1.50515
$w = 2100$	colog 8.49485 - 10
$(\phi + \omega) = 72^\circ 35'$	log 3.32222
57.3	colog 6.67778 - 10
$\phi = 30^\circ$	log 1.85944
	log 1.75815
	colog 8.24185 - 10
	lsec 0.06247
$D = 1006.8 \text{ yards}$		log 3.00292

THE DANGER SPACE (COLUMN 7)

914. The dimensions of a target evidently determine the limits within which the range must be known in order that the point of fall may occur within the limits of the target. The horizontal dimension of the target in the line of fire always constitutes a part of the allowance defined by these limits; if the target is a ship, this horizontal dimension is the beam of the ship when the latter is lying crosswise with respect to the line of fire, or the length of the ship when the latter is lying endwise. That part of the total range allowance which is defined by this

horizontal dimension, in no way depends upon the trajectory; it remains the same for all guns and at all ranges. The *height* of the target, however, contributes to this total range allowance an amount which depends on the angle of inclination of the trajectory near the point of fall, and which varies, therefore, according to both the gun and the range, and must be calculated accordingly.

The total range allowance with which we are dealing here may be thought of either as defining the limits within which the sight-bar range of the gun may vary, considering the target to remain at a fixed distance; or as defining the limits within which the distance to the target may vary, considering the sight-bar range to remain fixed. The amount of the allowance differs somewhat depending upon which of the above conceptions is applied in determining it, but the distinction is of no importance except at very short ranges, and need be considered only in connection with certain special problems which are met at Short Range Practice and which will be dealt with separately in Appendix B.

The total range allowance corresponding to the dimensions of the target, when considered in the sense of an allowable variation in target distance and of a fixed sight-bar range, is called the *danger space* and denoted by S (usually expressed in yards).^{*} According to this conception we may then state that the

Definition of danger space *danger space for a given target and trajectory is the greatest distance through which the target may be moved in the line of fire and still be intersected at some point by that trajectory.* This is illustrated in Figure 19, in which OMH is a trajectory whose point of fall is at the fixed range OH . A target, whose height is h and whose horizontal dimension in the line of fire is l , is shown in the position in which the trajectory strikes just at the waterline of

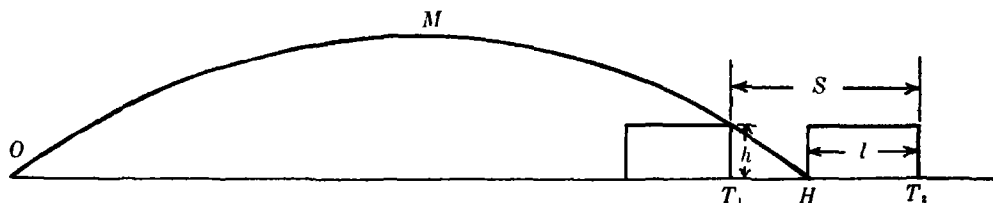


FIGURE 19

the side nearer the gun, and the position in which the same trajectory just fails to clear the upper edge of the side farther from the gun. The danger space, $S = T_1T_2$, is the distance between these two positions. From the construction of the figure, it is evident that the total danger space is composed of two parts, $HT_2 = l$ being the horizontal dimension in the line of fire, and T_1H being the projection of the height h upon the horizontal plane.

915. The value of that portion of the danger space which depends only on the height of the target (as T_1H in Figure 19), is tabulated in Column 7 of the range table; the value tabulated applies to a target height of 20 feet, and the value for any other heights (within reasonable limits) may be found therefrom by simple proportion (see also art. 920). In other words, *Column 7 of the range table gives the value of the danger space for a target whose height is 20 feet and whose horizontal dimension in the line of fire is zero.* It is to be understood, of course, that the usual range-table conditions apply here as elsewhere, and that the base of the target is there-

^{*} When taken in the opposite sense, i.e., as an allowable variation in sight-bar range at a fixed target distance, this quantity is termed the *hitting space*. The terms *danger space* and *hitting space* ordinarily are practically interchangeable, except at very short ranges.

fore considered to be in the horizontal plane which passes through the gun (art. 801).

In Figure 20, AB represents a target whose height is h and whose horizontal dimension in the line of fire is zero, and $S = AH$ is the danger space for the target against the trajectory OBH , whose horizontal range is $X = OH$, and maximum ordinate is y_0 . The danger space is thus seen to be the difference between the greatest distance, $X = OH$, and shortest distances, $x = OA$, at which the given

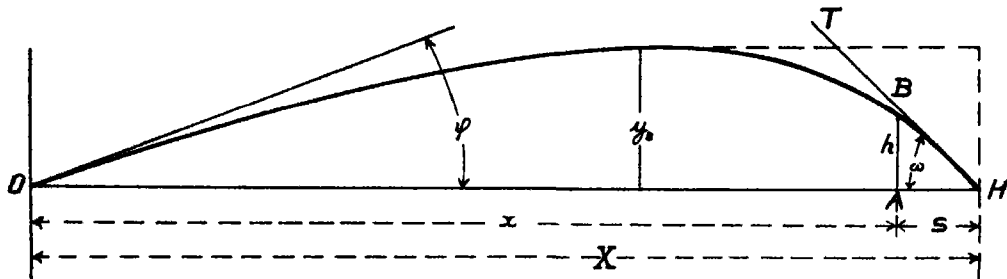


FIGURE 20

target may be situated and yet be intersected by the given trajectory. The direct solution for the danger space therefore involves the determination of the abscissa x of the trajectory corresponding to the ordinate $y = h$, whence $S = X - x$. A solution according to this principle may be made, of course, by finding the point x , $y = h$, by numerical integration. Satisfactory approximations of the danger space may be made, however, by methods which are much less laborious than the above

916. Except at very short ranges the value of h (Figure 20), within practical limits of target height, is small in comparison with the maximum ordinate y_0 , and the point in the trajectory where $y = h$ is near the point of fall. Considering that the trajectory, for the relatively short distance between the point $y = h$ and the point of fall, is practically a straight line which coincides with the tangent to the trajectory at the latter point, we have the approximate relation

Formula for
Column 7 for
moderate and
long ranges

$$S = h \cot \omega \quad (903)$$

in which ω is the angle of fall. The above formula gives a satisfactory approximation of the danger space (Col. 7) for ranges at which the angle of fall exceeds about 3° (about 4000–6000 yards for most naval guns). The application of (903) is illustrated in the following example.

Given: The 16" 2600 f.s. gun, range 10,000 yards, angle of fall $5^\circ 55'$.

Find: The danger space (range-table Column 7).

Since the target height for Column 7 is 20 feet, we shall use $h = 20$ feet. Then we have

$$\begin{array}{lcl} h = 20 \text{ feet} & \dots\dots\dots & \log 1.30103 \\ \omega = 5^\circ 55' & \dots\dots\dots & \text{c} \cot 0.98450 \\ S = 192.99 \text{ feet} & \dots\dots\dots & \log 2.28553 \\ & & = 64 \text{ yards} \end{array}$$

917. At shorter ranges the point $y = h$ (Fig. 20) may be so far from the point of fall that material error may result from considering the portion of the trajectory

Process for
computing
Column 7
for short
ranges

beyond this point to be a straight line. In such cases a solution may be made by a process that is somewhat indirect, but that is simple and sufficiently accurate for any practical purpose. In Figure 21 the trajectory OBH , whose horizontal range is OH , is shown just touching the top of the target AB , whose height is h and whose horizontal dimension in the line of fire is zero; then $S = AH$ is the danger space for the target height h at the range OH . There is shown also the trajectory whose point of fall is just at the bottom of the target AB , and whose horizontal range therefore is OA . The difference between the horizontal ranges, OH and OA , respectively, of these two trajectories, evidently is equal to the danger space for the longer of

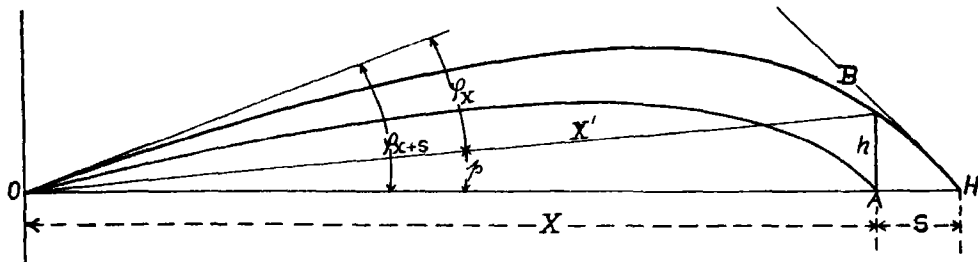


FIGURE 21

them. Let us assume, now, a value $OA = X$ for the horizontal range of the trajectory whose point of fall is at the bottom of the target, and find how much must be added to this to give the trajectory which just touches the top of the same target; this amount evidently equals the danger space S of the latter trajectory. The assumed range of the shorter trajectory being X , we then have the danger space S for the longer trajectory, i.e., for the range $X + S$.

Since h is always small in comparison with X , we may consider that the inclined range $X' = OB$ is equal to the horizontal range $X = OA$ (for example, at the shortest range tabulated, 1000 yards, and for a 40-foot target, which is as high as we have occasion to consider, the difference between OB and OA is about 0.1 yard). Also, the angle of position $BOA = p$ is always small in any practical case (for example, $p = 46'$ in the rather extreme case just cited). Under these conditions the theory of rigidity of the trajectory is applicable (art. 319), and we may consider that the trajectory OB represents the trajectory OA tilted upward by the amount of the angle p . Since the two trajectories are considered to be similar in all respects, the angle of elevation of the trajectory OB is equal to the angle of departure of the trajectory OA , i.e., to the angle of departure corresponding to the horizontal range X . We shall therefore denote the angle of elevation of trajectory OB by ϕ_x , as illustrated. Likewise, since the horizontal range of the trajectory OH equals $X + S$, we shall denote its angle of departure by ϕ_{x+s} , i.e., the angle of departure corresponding to the horizontal range $X + S$.

From the figure it is apparent that the angle p is defined by the relation

$$\tan p = \frac{h}{X} \quad (904)$$

whence, with a given range X and target height h , p may be found. From Columns 1 and 2 of the range table, the value of ϕ corresponding to any value of X may be found, and hence the value of ϕ_x is available directly. As shown in the figure,

$$\phi_{x+s} = \phi_x + p \quad (905)$$

and with ϕ_{X+S} we can find in the range table the corresponding range, which, in this case, is $X+S$. The required danger space is then

$$S = (X + S) - X. \quad (906)$$

It is important to note, however, that the value of S so found applies not to the assumed range X , but to the range $X+S$. It is for this reason that the process is indirect. A direct solution according to the same principle is not possible without resort to successive approximations. For if we seek to determine the value of S corresponding directly to an assumed value of X , we cannot find p , since in this case we

must take $\tan p = \frac{h}{X-S}$. The process as actually used also requires successive

approximations to be made, insofar as the solution for the danger space corresponding to a particular range is concerned. In the computations for range-table values, however, this is of no consequence, since we make solutions for many ranges, plot the values of danger space against the ranges to which they apply, and then take from the graph the values of the danger space corresponding to the tabular values of range.

918. For further demonstration of the process outlined above, let us apply it to the case of the 16" 2600 f.s. gun, and assume $X = 1350$ yards = 4050 feet.

Examples of computation of Column 7 for short ranges From the 16" 2600 f.s. range table we find that at 1350 yards the angle of departure is $34'.0$; hence we have $\phi_X = 34'.0$. Since Column 7 of the range table is based on a target height of 20 feet, we have $h = 20$ feet, whence, from (904),

$$\tan p = \frac{20}{4050}, \quad \text{and} \quad p = 17'.0.$$

Also, from (905),

$$\phi_{X+S} = 34'.0 + 17'.0 = 51'.0$$

and from the range table we find that this angle of departure corresponds to the range 2004 yards, or $X+S = 2004$ yards. Then, from (906),

$$S = 2004 - 1350 = \underline{654 \text{ yards}}.$$

The result thus obtained is, that for a 20-foot target the danger space is 654 yards at the range 2004 yards; in other words, the point of fall of the given trajectory being at a distance of 2004 yards from the gun, a 20-foot target can be moved from this point of fall toward the gun a distance of 654 yards, or to a point only 1350 yards from the gun, before its top falls below the trajectory.

Now let us assume $X = 2000$ yards = 6000 feet, and $h = 20$ feet. Then

$$\tan p = \frac{20}{6000}, \quad \text{and} \quad p = 11'.5$$

$$\phi_{X+S} = 50'.9 + 11'.5 = 1^\circ 02'.4$$

$$X + S = 2430 \text{ yards}$$

$$S = 2430 - 2000 = \underline{430 \text{ yards}}$$

whence we find that at the range 2430 yards the danger space for the 20-foot target is 430 yards.*

919. It will be observed in Figure 20 that if the height of the target is equal to or greater than the maximum ordinate, then the target may be moved the entire distance from the point of fall to the gun, i.e., the danger space is equal to the whole range. The maximum range at which this occurs, for a given height of target, is called the *danger range*, and it is evidently the range at which the maximum ordinate just equals the given height of target. It will be noted that in the range table the values in Columns 1 and 7 are the same whenever the value in Column 8 (maximum ordinate) is 20 feet or less. It will be clear that if the danger range, or any range less than the latter, is set on the sights of a gun, and the gun is sighted at the bottom of the target, then the target may be located at any distance from the gun equal to or less than the range set on the gun, and yet be hit. This condition holds true, however, only if the gun and target are in the same horizontal plane, as is assumed for the range table. Actually, the values in Column 7 which are equal to the whole range have no practical value, since the gun is in fact always above the water. For the same reason, other values from Column 7 are not applicable directly to the usual situation at short ranges, in which the gun's height above the water is comparable to the height of the maximum ordinate itself.

920. At moderate and long ranges, the values of the danger space (Col. 7) for target heights other than 20 feet, within the limits of the latter that are likely to occur in practice, may be found by applying simple proportion to the values tabulated in Column 7. At short ranges, however, very great errors may result from such a process. For example, in the 16"2600 f.s. range table, at the range 1300 yards, the maximum ordinate is 8 feet, and the danger space for a 20-foot target (Col. 7) is therefore 1300 yards. This value evidently applies also to any target height equal to or greater than 8 feet; but for a target height the least bit less than 8 feet the danger space at once decreases to a value less than one-half as great as the above. Again, at 2600 yards the danger space for the 20-foot target is 368 yards, but for a 40-foot target it is 2600 yards, since the maximum ordinate at this range is just 40 feet. It is evident, then, that proportionality between danger spaces for the same trajectory, with respect to their corresponding target heights, vanishes altogether when either of the target heights involved equals or exceeds the maximum ordinate of the given trajectory. But even for target heights that remain within the limits of the maxi-

* The process just outlined is the one now in use for computing Column 7. For range tables computed before about 1926, a very much less accurate process was employed in connection with Column 7. The formula previously used for short ranges was,

$$S = h \cot \omega \left(1 + \frac{h \cot \omega}{X} \right).$$

Since this formula is now obsolete, its derivation will not be given here, but it may be found in earlier editions of this book. It is worthy of note, however, that in many of the range tables still in current use at this writing, Column 7 represents values found by the above formula. This is also the case with all of the tables given in *Range and Ballistic Tables, 1935*, which accounts for the discrepancies between the values found above and those given in the 16"2600 f.s. range table. This table gives $S = 527$ yards at range 2004 yards, and $S = 402$ yards at range 2430 yards, whereas the results found above are, respectively, 654 yards and 430 yards. It will be observed that the discrepancy is very considerable at the shorter range but much less serious at the longer range. Above about 3000 yards the difference becomes immaterial; this applies to all guns. The 16"2600 f.s. range table in current use in the fleet contains the values as found by the more accurate process given above.

imum ordinate, the use of simple proportion in connection with Column 7 is subject to errors which increase as either of the target heights involved approaches the height of the maximum ordinate; this is very likely to be the case at short ranges, where ordinary target heights are comparable to the heights of the maximum ordinates. For the reasons outlined in the present and foregoing paragraphs it must be concluded, therefore, that the values in Column 7 are practically useless at short ranges. Methods for obtaining values that are useful in connection with certain special short-range problems will be taken up in Appendix B.

CHANGE IN HEIGHT OF IMPACT (COLUMN 19)

921. Column 19 gives the change in height of impact that results from a variation of 100 yards in the sight-bar range. Such information is useful, for example, in direct-flight spotting, when the sight-bar range is adjusted by observing the point of impact in the vertical plane of the target. Let us suppose that the situation is as illustrated in Figure 21, and that the point of impact is observed to be at a height h feet above the waterline of the target AB , the sight-bar range set on the gun being equal to OH . If this point of impact is to be lowered to the waterline, the sight-bar range evidently must be reduced to OA . The problem is similar to that of finding the danger space for a given target height. In the latter case we found the change in range corresponding to a given height h ; in the present case we are to find the height h that corresponds to a given change of range, 100 yards being the amount of the change assumed for Column 19.

We shall denote the required change in height of impact by Δh , and the change of range to which it corresponds by ΔX . Since $\Delta X = 100$ yards, which is small in comparison with the whole range X in any practical case, we may assume without material error that the portion of the trajectory from the point where $y = \Delta h$ to the point of fall is a straight line, and that

Formula for
Column 19 of
the range table

$$\Delta h = \pm \Delta X \tan \omega \quad (907)$$

ω being the angle of fall at the assumed range X . The sign \pm has been introduced in order to indicate that the formula is applicable both for increases and for decreases in X , i.e., for $\Delta X = \pm 100$ yards. From Figure 21 it will appear that for the case of the increase in X , the value of ω should be found with the range $X + \Delta X$, rather than with X directly. However, the change in $\tan \omega$ corresponding to a change of 100 yards in range is sufficiently small, in any case, to have an inappreciable effect on the value of Δh that is to be found, and we may therefore use (907) with the value of ω corresponding to the assumed X whether we are dealing with an increase or a decrease in the latter. The application of (907) is illustrated in the following example.

Given: The 16"2600 f.s. gun, range 10,000 yards, angle of fall $5^\circ 55'$.

Find: The change in height of impact corresponding to a variation of 100 yards in sight-bar range.

$$\begin{aligned} \Delta X &= \pm 300 \text{ feet} \dots\dots\dots \pm \log 2.47712 \\ \omega &= 5^\circ 55' \dots\dots\dots \tan 9.01550 - 10 \\ \Delta h &= \pm 31 \text{ feet} \dots\dots\dots \pm \log 1.49262 \end{aligned}$$

The result obtained means that, for the 16"2600 f.s. gun at the range 10,000 yards, an increase or decrease of 100 yards in the sight-bar range of the gun will cause the point of impact in the vertical plane of the target to be raised or lowered,

respectively, 31 feet. It will be observed that the same information is available from Column 7, for which the computation has been made in article 916. Since, from the latter, a target height of 20 feet corresponds to a

Relation
between
Columns
7 and 19

change of 64 yards in range, we may deduce that the height $\frac{100}{64} \times 20$

= 31 feet corresponds to a change of 100 yards. This relation between Columns 7 and 19 exists at all but very short ranges.

EXERCISES

1. *Given:* The diameter and weight of the projectile, the range, angle of departure, angle of fall, drift coefficient D' and final twist of rifling μ .

Find: The values for Column 6 (drift) of the range table for the given ranges.

		Given							Answers.
		d	w	X (yards)	ϕ	ω	D'	μ	D (yards)
A	5"-3150 f.s.	5	50	5000	2° 01'	2° 52'	.7869	25	9.1
B		5	50	8000	4 14	7 31	.7869	25	35.1
C		5	50	10000	6 26	12 48	.7869	25	72 0
D		5	50	15000	14 59	30 52	.7869	25	264.8
E	16"-2600 f.s.	16	2100	3000	1 18	1 22	.67	32	2.8
F		16	2100	5000	2 14	2 27	.67	32	8.2
G		16	2100	8000	3 49	4 23	.67	32	23.1
H		16	2100	12000	6 11	7 40	.67	32	58.7
I		16	2100	20000	12 15	17 10	.67	32	211.3

2. *Given:* The range and the angle of fall.

Find: The values for Column 7 of the range table for the given ranges (danger space for a 20-foot target). (Use formula 903.)

		Given		Answers
		X (yards)	ω	S (yards)
A	5"-3150 f.s.	5000	2° 52'	133
B		8000	7 31	51
C		10000	12 48	29
D		15000	30 52	11
E		22000	57 33	4
F	16"-2600 f.s.	5000	2 27	156
G		12000	7 40	50
H		20000	17 10	22
I		39000	52 53	5

3. *Given:* The range and the angle of fall.

Find: The values for Column 19 of the range table for the given ranges (change in height of impact for a variation of 100 yards in the sight-bar range).

		Given		Answers
		X (yards)	ω	Δh (Col 19) (feet)
A	5"-3150 f.s.	5000	2° 52'	15
B		8000	7 31	40
C		10000	12 48	68
D		15000	30 52	179
E		22000	57 33	472
F	16"-2600 f.s.	3000	1 22	7
G		8000	4 23	23
H		12000	7 40	40
I		15000	10 44	57
J		20000	17 10	93

4. Proceeding as illustrated in article 918, make computations for S with the ranges 1600 yards and 1800 yards (16"-2600 f.s. gun) and for a target height of 20 feet. Using the two values of danger space (Col. 7) and their corresponding ranges as thus determined, and using also the two sets of values already found in article 918, plot a graph of danger space against range (scale 1" = 100 yards), and from this graph determine the values of the danger space for the ranges 2000, 2100, 2200, 2300, and 2400 yards.

Answers (see footnote on p. 119):

Range (yds.)	Column 7 (yds.)	Range (yds.)	Column 7 (yds.)
2004	654	2000	657
2148	548	2100	580
2281	481	2200	518
2430	430	2300	471
		2400	440

CHAPTER 10

THE DETERMINATION OF THE EFFECTS OF VARIATIONS FROM RANGE-TABLE STANDARDS FOR INITIAL VELOCITY, WEIGHT OF PROJECTILE, AND ATMOSPHERIC DENSITY (RANGE-TABLE COLUMNS 10, 11, and 12).

New Symbols Introduced

- ΔX_V . . . The change in range due to a variation in the initial velocity.
- ΔX_C . . . The change in range due to a variation in the ballistic coefficient (or in any factor contained in the latter, such as δ and w .)
- M . . . A multiplier to be used with Column 12 of the range table (art. 1014).
- ΔV_w . . . The change in initial velocity due a variation in the weight of the projectile.
- ΔC_w . . . The change in the ballistic coefficient due to a variation in the weight of the projectile.
- ΔX_w . . . The change in range due to a variation in the weight of the projectile.
- m . . . A coefficient used in the formula for finding ΔV_w (1003).

1001. Having examined the problem of determining the values of elements of the trajectory under the standard conditions assumed for range tables (art. 801), our next problem is to consider the means by which these values may be adjusted for variations from the assumed standards. Although such variations affect all elements of the trajectory, our chief concern is to determine their effects on the location of the point of fall. The changes in range due to variations in initial velocity, weight of projectile, and atmospheric density, and of wind, motion of gun, and motion of target in the line of fire; and the lateral deviations due to wind, motion of gun, and motion of target across the line of fire; are given in Columns 10-18 of the range table. In the present chapter we shall deal with Columns 10, 11, and 12, which give the changes in range due to variations, respectively, in initial velocity, weight of projectile, and atmospheric density.

The effect of a variation from standard in any of the factors on which the trajectory depends, evidently can be found by making a complete solution of the

General principles involved in determining effects of variations from standard conditions

trajectory for the non-standard combination, and comparing it with the solution corresponding to the standard combination. For example, if the change (ΔX_V) in range, corresponding to a reduction (ΔV) from the standard initial velocity, is to be found, then ΔX_V evidently is the difference between the ranges corresponding to the combinations ϕ, V, C , and $\phi, (V - \Delta V), C$, the values of ϕ and C being the same for both cases. Similarly, the variation in range corresponding to a variation from standard in any of the factors contained in C can be found by comparing solutions made with the standard and non-standard values of C , the values of ϕ and V remaining the same. This direct method is particularly advantageous for determining variations in the terminal elements, since the values of the latter, for any desired combination of ϕ, V , and

C , can be found readily in ballistic tables. We shall therefore use this method in finding the values for Columns 10, 11, and 12.*

CHANGE IN RANGE DUE TO A GIVEN VARIATION IN INITIAL VELOCITY (COLUMN 10).

1002. Practice has varied as to the size of velocity variation assumed for the tabular values in Column 10. In some of the older tables that are still in use, the value in Column 10 corresponds to a 50 f.s. variation in initial velocity, but the later practice is to tabulate in this column values that correspond to a 10 f.s. variation. The heading of Column 10 indicates in all cases the size of variation to which the tabulated values apply. Practice has varied also as to the sign of the velocity variation assumed in deriving the values for this column. Formerly it was the practice to make these values equally applicable either to increases or to decreases of velocity. This was done by taking the mean of values found, respectively, with plus and minus variations in V . The more recent practice is to base Column 10 only on a minus variation in V , since variations from the standard V are practically always of this sign. Erosion, of course, always causes reductions in V ; variations in V due to non-standard powder temperature also are reductions in practically all cases, since the standard powder temperature, 90°F., is also the highest temperature permitted in magazines.

The heading of Column 10 in the more recent tables states that the values in this column are the changes in range corresponding to a (+) 10 f.s. variation in initial velocity. This is not intended to imply that the values were *derived* on the basis of a plus variation only, but merely to indicate that the sign of the tabulated values corresponds to an increase of initial velocity (the plus sign is to be understood when no sign appears before the tabulated values). No confusion should arise as to the proper sign to be used with these values, since an increase in V always causes an increase in range, and a decrease in V always a decrease in range. Information as to the practice followed in deriving the values in this column may be found in the introductory pages of the range table. All of the extracts included in *Range and Ballistic Tables, 19'5*, are from the older range tables, in which Column 10 is based on \pm variations in V .

* By partial differentiation of the fundamental equations of the trajectory, with respect to each of the several factors (ϕ , V , C) on which the latter depends, it is possible to set up expressions which define the differential relations between any of these factors and any element of the trajectory. By differentiating the fundamental equations with respect to C alone, the differential relation between C and any element is found, and hence it is possible, for example, to express the relation between a variation in C and the effect of the latter on the values of x , y , t , v , θ , at any point in the trajectory. By integrating these differential expressions, it is then possible to evaluate these effects within any desired limits. Thus the effect of a variation in s (and hence in C) which is assumed to apply only within specified limits of altitude, can be found by integrating the appropriate differential expression between these limits. Practical applications of variations of this character will be dealt with presently, and reference will be made, in this connection, to the principles involved in the determination of such variations by the method just outlined, i.e., by the method of differential variations. A complete demonstration of this method is, however, much too lengthy to be included in this text. Comprehensive treatments of the differential-variation method are available in the following sources: *A Course in Exterior Ballistics* (War Department Document No. 1051, December, 1920), by R. S. Hoar; *The Method of Numerical Integration in Exterior Ballistics* (War Department Document No. 984, October, 1919), by D. Jackson; *New Methods in Exterior Ballistics*, by F. R. Moulton

1003. The following example illustrates the later practice that is followed in the computation of values for Column 10. This example is based on the same gun and angle of departure assumed for the example given in article 815.

Given: The 16"2600 f.s. gun, $\phi = 25^\circ$, $\text{Log } C = 1.12707$.

Find: The change in range due to a variation of $(-)$ 10 f.s. in the initial velocity.

We shall find the ranges, X_1 and X_2 , corresponding to the velocities, respectively, $V_1 = 2600$ f.s. and $V_2 = 2590$ f.s., using the A.L.V.F. Tables, with $\phi = 25^\circ$ and $\text{Log } C = 1.12707$ in each case. The difference between these ranges, which we shall denote by ΔX_V , then represents the change in range due to the given variation in V . Converting the velocities to metric units, and then entering the A.L.V.F. Tables, we have

$V_1 = 2600$ f.s.	$\log 3.41497$	$V_2 = 2590$ f.s.	$\log 3.41330$
(art. 705)	$\log 9.48402 - 10$	(art. 705)	$\log 9.48402 - 10$
$V_1 = 792.49$ m.s.	$\log 2.89899$	$V_2 = 789.44$ m.s.	$\log 2.89732$

From the table headed $\phi = 25^\circ$ Range

For $V = 790$,	$X = 28,069 + .604 \times 538 = 28,394$
For $V = 800$,	$X = 28,601 + .604 \times 554 = 28,936$
For $V_1 = 792.5$,	$X_1 = 28,394 + .25 \times 542 = 28,530$ m.
For $V = 780$,	$X = 27,541 + .604 \times 522 = 27,856$
For $V = 790$,	$X = 28,069 + .604 \times 538 = 28,394$
For $V_2 = 789.4$,	$X_2 = 27,856 + .94 \times 538 = 28,362$ m.

$\Delta X_V = (-)$ 168 m.	$\log 2.22531$
(art. 705)	$\log 0.03886$

$\Delta X_V = (-)$ 184 yards.	$\log 2.26417$
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This result differs only slightly from that given in Column 10 of the 16"2600 f.s. range table (*Range and Ballistic Tables, 1935*), the latter having been found by assuming a (\pm) 10 f.s. variation.

1004. The differences with respect to V , as tabulated in the A.L.V.F. Tables for range, evidently are the values of ΔX_V , in meters, corresponding to successive variations of 10 m.s. in V . By examining these differences we may readily determine what degree of error is occasioned by using simple proportion in connection with Column 10 to find the changes in range corresponding to velocity reductions considerably in excess of 10 f.s. For example, let us assume $\phi = 40^\circ$, $\text{Log } C = 1.135$, and a standard V of 790 m.s. In this case we find, from the A.L.V.F. Tables, that the change of range corresponding to a reduction of 10 m.s. (i.e., from 790 m.s. to 780 m.s.) is 711 m., while the change of range corresponding to a reduction of 30 m.s. (i.e., from 790 m.s. to 760 m.s.) is 2109 m. If simple proportion is applied to the change for 10 m.s. in order to find the change for 30 m.s., we then find for the latter the value $3 \times 711 = 2133$ m., which is in error by 24 m., or by a little more than 1% with respect to the correct value of ΔX_V ; in comparison with the range (about 35,000 m.) the difference of 24 m. amounts to an error of less than one-tenth of one percent.*

* For some range tables the value of ΔX_V for $(-)$ 10 f.s., as tabulated in Column 10, has been derived by taking one-tenth of the value of ΔX_V for $(-)$ 100 f.s. In such cases the situation discussed above is reversed, i.e., changes of range determined by applying simple proportion to the values given in Column 10 are somewhat more accurate for large velocity changes than for small ones.

The case just considered may be regarded as a rather extreme test as to the degree of error that may be incurred by applying simple proportion to Column 10 values; for smaller values of ϕ and C , the degree of error becomes relatively even less. The comparison between 10 m.s. and 30 m.s. values, as made above, serves to indicate very nearly what the situation is with respect to 10 f.s. and 100 f.s. values. The fact that the ratio in the latter case is increased to 10 to 1, as compared with only 3 to 1 in the former case, is of no consequence, since the 10 f.s. value is in fact obtained by linear interpolation with respect to a 10 m.s. tabular interval. In other words, no greater inaccuracy is incurred by deriving the 100 f.s. value from the 10 f.s. value, than by deriving the 30 m.s. value from the 10 m.s. value.

Velocity reductions due to the combined effects of erosion and non-standard powder temperature ordinarily do not exceed 100 f.s., and rarely exceed 125 f.s. It may be concluded, therefore, that the process of applying simple proportion to Column 10 values for finding changes in range corresponding to any reductions in initial velocity that are likely to be encountered in practice, does not of itself entail any appreciable loss of accuracy.

1005. In all of the above determinations of the effect of velocity variations, it has been assumed, however, that the value of ΔX_V depends only on the variation in velocity itself. That is to say, the values of ΔX_V , for all velocity reductions considered, have been based on the same value of C . The situation is altered somewhat by the consideration that i (and hence C) actually varies with the velocity. Such variations in i , if not accounted for, cause the values of ΔX_V to be in error by amounts which increase with the size of velocity variations to which the values of ΔX_V apply; this will remain true whether ΔX_V is derived from Column 10 or directly from the ballistic tables. Such errors can be eliminated only by determining ΔX_V experimentally, and this procedure has been followed in some cases.

1006. For some target practices, the initial velocity of the gun is very much less than the service initial velocity; for example, the target-practice velocity of the 16"/45 gun is 2000 f.s., or 600 f.s. less than the service velocity of 2600 f.s., and in the case of the 5"/51 gun the difference is 850 f.s. (3150-2300). These reductions in initial velocity are much too great to be handled by means of Column 10, and additional range tables are therefore prepared for the target-practice velocities; in some cases other special velocities also are provided for by means of additional range tables. A separate experimental ranging is conducted for each of these additional range tables, and changes in the value of i are thus fully accounted for. All other operations involved in the preparation of a range table are also performed separately for each of the tables.

CHANGE IN RANGE DUE TO A GIVEN VARIATION IN ATMOSPHERIC DENSITY (COLUMN 12).

1007. In all U. S. Navy range tables, Column 12 is based on a variation of $(\pm) 10\%$ in atmospheric density. The heading of Column 12 in the more recent tables states that the values in this column are the changes in range corresponding to a $(-)$ 10% variation in atmospheric density. This is not intended to imply that the values were derived on the basis of a minus variation only, but merely to indicate that the sign of the tabulated values corresponds to a decrease in density (the plus sign is to be understood when no sign appears before the tabulated values). No con-

fusion should arise as to the proper sign to be used with these values, since an increase in atmospheric density always causes a decrease in range, and a decrease in atmospheric density always causes an increase in range.

1008. It is shown by formulas (404) and (406), respectively, that a given $\pm\%$ variation in atmospheric density causes an equal $\pm\%$ variation in the density factor δ , and hence an equal $\mp\%$ variation in C . The change in range due to a (\pm) 10% variation in atmospheric density therefore can be found by determining the change in range due to a (\mp) 10% variation in C . The problem is handled very simply, in accordance with the same general principle that has already been applied in connection with variations in V , i.e., by interpolation in the A.L.V.F. Tables. The process is illustrated in the following example, which is based on the same gun and angle of departure assumed for the example given in article 815.

Given: The 16"2600 f.s. gun, $\phi = 25^\circ$, $\text{Log } C = 1.12707$.

Find: The change in range due to a variation of (\pm) 10% in atmospheric density.

We shall find the ranges corresponding to values of C decreased and increased by 10% with respect to the standard value of C , using the A.L.V.F. Tables, with $\phi = 25^\circ$ and $V = 792.5$ m.s. (2600 f.s.) in each case. The difference between these ranges represents the change of range corresponding to a change of 20% in C , i.e., the sum of the changes of range due to a 10% decrease in C and to a 10% increase in C . One half of this sum then is the *mean* of the changes in range due to a 10% decrease and a 10% increase in C , and it may be termed the change in range due to a (\mp) 10% variation in C , which corresponds to a (\pm) 10% variation in atmospheric density. We shall let the symbol ΔX_C denote a change in range due to a change in C , and in this case specifically a change in range due to a change in atmospheric density.

Proceeding as just outlined, we find the reduced and increased values of C as follows.

C	log 1.12707..	log 1.12707
.90.....	log 9.95424 - 10	
1.10.....		log 0.04139
C_1	log 1.08131	
C_2		log 1.16846

From the table headed $\phi = 25^\circ$ Range

For Log $C = 1.075$,	$X = 26,988 + \frac{1}{4} \times 499 = 27,113$
For Log $C = 1.095$,	$X = 27,529 + \frac{1}{4} \times 516 = 27,658$
For Log $C_1 = 1.08131$,	$X_1 = 27,113 + .316 \times 545 = 27,285$ m.
For Log $C = 1.155$,	$X = 29,143 + \frac{1}{4} \times 565 = 29,284$
For Log $C = 1.175$,	$X = 29,679 + \frac{1}{4} \times 581 = 29,824$
For Log $C_2 = 1.16846$,	$X_2 = 29,284 + .673 \times 540 = 29,647$ m.

$$\Delta X_C = (\mp) \frac{1}{2} (29,647 - 27,285) = 1181 \text{ m.}$$

$\Delta X_C = (\mp) 1181$ m.....	log 3.07225
(art. 705).....	log 0.03886
$\Delta X_C = (\mp) 1292$ yards*.....	log 3.11111

* This agrees with the value given in the latest 16"2600 f.s. range table, which is not available for publication in *Range and Ballistic Tables, 1935*. The range tables given in the latter volume are obsolete, which accounts for many of the differences noted between results given in these tables and results found according to the computations given in this text.

1009. As has been noted in the procedure by which the above value of ΔX_C has been found, the latter represents the mean of the changes in range due to a 10% decrease and a 10% increase in C . The range corresponding to the standard

Degree of accuracy afforded by Column 12 C of the above problem has already been found to be 28,530 m. (art. 815); the ranges corresponding to a 10% decrease and a 10% increase in C are, respectively, 27,285 m. and 29,647 m. (art. 1008). The change in range for the 10% decrease in C therefore is $28,530 - 27,285 = 1245$ m., and for the 10% increase it is $29,647 - 28,530 = 1117$ m. If the mean of these (1181 m.) is used for the value of Column 12, the latter is then 64 m. or about 5% too small for the 10% decrease in C , and 64 m. or about 6% too great for the 10% increase in C ; in comparison with the whole range, however, the error in either case is only about 0.2%. The situation is substantially the same at greater and lesser ranges, i.e., the values in Column 12 are somewhat too small for decreases in C and somewhat too great for increases in C . It follows that, with respect to variations in atmospheric density, the values in Column 12 are somewhat too small for increases in density, and somewhat too great for decreases in density.

In actual practice, especially at sea, variations from the standard atmospheric density rarely amount to as much as 10%, and the degree of inaccuracy involved in the use of Column 12, as noted above, therefore represents rather wide extremes. Ordinarily variations in atmospheric density are confined within comparatively narrow limits, and the errors incident to the use of Column 12 are limited accordingly. In any event, it is probable that the degree of inaccuracy involved in the use of Column 12 is always relatively small in comparison with the degree of inaccuracy involved in the practical determination of the atmospheric density itself, since the latter, at best, is always subject to rather broad approximations (art. 425).

1010. From what already has been said with regard to the character of variations in atmospheric density (art. 420), it is apparent that methods must be devised for discriminating between variations that pertain to the entire trajectory and those that apply only to portions of the trajectory. In other words, it is necessary to devise means for taking into account a given variation in atmospheric density that pertains only within specified limits of altitude, or, more generally, variations of different magnitude that apply within different limits of altitude.

Method of handling density variations that are not uniform throughout the trajectory

This problem can be handled by expressing the differential relation between X and C , and by integrating this expression within the required limits (see foot-note adjacent to article 1001). It is possible, in this manner, to find the effect on the trajectory as a whole of a variation in C , and hence in atmospheric density, that applies only within specified limits of the ordinates of that trajectory. That is to say, a value of ΔX_C can be found that expresses the change in range of the whole trajectory due to the effect of a (\pm) 10% variation in atmospheric density on the portion of a trajectory which is included within any given zone of altitude, as for example for zones extending from the surface to a height of 600 feet, from the height 600 feet to the height 1500 feet, from 1500 feet to 3000 feet, etc.

The practical method of dealing with this problem is to compare the effects of a given density variation operating only on a limited portion of the trajectory, with the effect of an equal density variation operating on the entire trajectory, and to set up ratios accordingly. In this manner the need for additional range-table columns is avoided. For example, let us assume that in the case of the same trajectory for which we have already found the value of Column 12 (art. 1008),

it has been determined that the effect of a 10% variation in atmospheric density operating only on those portions of the trajectory which are included within the limits of the zone 0-600 feet, amounts to only .04 of the effect a 10% variation operating throughout the entire trajectory. The amount of the latter being the value tabulated in Column 12, in this case 1292 yards, the effect of the 10% variation which is confined to the zone 0-600 feet then is $.04 \times 1292 = 52$ yards.

1011. The ratios referred to above are called *air-density weighting factors*, since they express the weights of the effects of density variations confined within specified limits of the trajectory, relatively to the effects of equal variations extending to the entire trajectory. The values of the weighting factors for a given trajectory evidently depend upon the characteristics of the trajectory itself, as well as upon the sizes of the zones to which the factors are to apply. However, the establishment and use of a separate series of such factors, for each of the great many trajectories included in the range tables of the various guns that are in use, would involve a degree of elaboration that is hardly warranted by the practical limitations of accuracy that apply to the measurement of aloft densities themselves. It is the present practice, therefore, to use for surface fire a single table of air-density weighting factors, which is based on a standard series of altitude zones and on a classification of trajectories only as to the number of such zones included within the limits of their respective maximum ordinates. The values of the weighting factors contained in this table represent averages deduced from a comprehensive analysis of such factors for all trajectories (i.e., including all guns) that are to be served by the table. Despite the wide limits assumed in obtaining these average factors, the latter afford a degree of accuracy that is commensurate with the accuracy of the density determinations themselves.

1012. The following table of air-density weighting factors is based on an analysis made at the Aberdeen Proving Ground according to the principles outlined above.* The arguments to be used in entering this table are the maximum ordinate of the trajectory (vertical argument), and the limits of altitude (horizontal argument) to which the weighting factor applies. It will be observed that the horizontal argument represents zones of altitude; these zones have been accepted as standard in connection with the measurement of aloft densities. The arrangement and use of the table can best be explained by examples.

Let us assume a trajectory whose maximum ordinate is 9,000 feet. We find

* This table is based on the formula

$$1 - p = 0.48 (1 - k)^{1/2} + 0.52 (1 - k)^{3/2}$$

in which p denotes the weighting factor, and k denotes the ratio y/y_0 to which the weighting factor pertains. For example, to find the weighting factor for the portions of the trajectory included between the surface and a height equal to one-half of the maximum ordinate, the value $k = \frac{1}{2}$ is substituted in the formula and the value of p is found to be .48. To find the weighting factor for the portions of a trajectory included between heights equal, respectively, to one-half and three-quarters that of the maximum ordinate, we find the difference between the values of p corresponding, respectively, to $k = \frac{1}{2}$ and $k = \frac{3}{4}$, which results, in this case, in the value $p = .69 - .48 = .21$. Further details relative to the derivation of these weighting factors are given in Chapter XV, *A Course in Exterior Ballistics* (War Department Document No. 1051, December, 1920), by R. S. Hoar, and on pp. 237-243, *Computation of Firing Tables for the U. S. Army*, by H. P. Hitchcock.

The U. S. Army now uses two sets of air-density weighting factors, one for ordinary terrestrial fire and one for antiaircraft and other high-angle fire. These weighting factors, and the corresponding ballistic-density tables, are given in Tables IX to XII, *Technical Regulations No. 1236-1* (U. S. War Department, June, 1934).

Air-density Weighting Factors

Maximum ordinate (feet)	Zones (feet)										
	0-600	600-1,500	1,500-3,000	3,000-4,500	4,500-6,000	6,000-9,000	9,000-12,000	12,000-15,000	15,000-18,000	18,000-24,000	24,000-30,000
600	1.00
1,500	.39	0.61
3,000	.20	.28	0.52
4,500	.18	.19	.31	0.37
6,000	.10	.14	.24	.21	0.31
9,000	.06	.10	.16	.16	.15	0.37
12,000	.05	.07	.12	.12	.12	.21	0.31
15,000	.04	.06	.10	.10	.09	.18	.16	0.27
18,000	.04	.04	.08	.08	.08	.16	.15	.14	0.23
24,000	.03	.03	.06	.06	.06	.12	.12	.11	.10	0.31
30,000	.02	.03	.05	.05	.05	.10	.09	.09	.09	.16	0.27

tabulated against the vertical argument 9,000 and the horizontal argument 0-600, the value .06. This means that for a trajectory whose maximum ordinate is 9,000 feet, the weighting factor for the portions of the trajectory included between the surface and a height of 600 feet, is .06; and further, that for this trajectory the change in range due to a 10% variation in atmospheric density which is confined to the zone 0-600 feet, is equal to .06 of the change in range due to a 10% variation in density that extends throughout the entire trajectory (i.e., .06 of the value of Column 12 for that trajectory). Similarly, against the vertical argument 9,000 and the horizontal argument 6,000-9,000, we find the weighting factor .37, which means that, for the same trajectory, the change in range due to a 10% variation in atmospheric density which is confined to the zone 6,000-9,000 feet, is equal to .37 of the value of Column 12 for that trajectory. The complete series of weighting factors for the same trajectory is as given in the following table.

Zone (feet)	Weighting factor	Change in range due to 10% variation in air density (yds.)
0-600	.06	60
600-1,500	.10	100
1,500-3,000	.16	160
3,000-4,500	.16	160
4,500-6,000	.15	150
6,000-9,000	.37	370
	1.00	1000

Assuming that the value of Column 12 for this trajectory is 1000 yards, the changes in range due to a 10% variation in air density confined to the several zones included within the trajectory, are as shown in the right-hand column of the above table. By the same process, we can apply a *different* density variation in each zone and find the corresponding change in range for each zone, in which case the sum of these changes represents the aggregate effect on the whole trajectory of the several different density variations, each duly weighted according to its own zone.

1013. In practice it is more convenient to apply the weighting factors directly to the density factors of the several zones, and thus to deduce a weighted mean

Use of air-density weighting factors for determining ballistic density

density factor, or *ballistic density*, for the entire trajectory (art. 421). For example, let us assume density factors for the various zones as shown in the following table; it is to be understood that each of these density factors is the ratio of the actual density

observed in that zone to the standard density for the same zone. (The values given are not to be considered as typical; round numbers have been chosen for convenience in following this elementary example.) Let us also assume the same trajectory for which the weighting factors have already been found in the foregoing article, and find the ballistic density for this trajectory.

Zone (feet)	Observed density factor	Weighting factor	Weighted density factor
0-600	1.060	.06	.0636
600-1,500	1.050	.10	.1050
1,500-3,000	1.040	.16	.1664
3,000-4,500	1.030	.16	.1648
4,500-6,000	1.020	.15	.1530
6,000-9,000	1.010	.37	.3737
			<hr/> 1.0265

The ballistic density for this trajectory is therefore 1.026, or, in other words, the weighted mean variation in atmospheric density for the entire trajectory, with respect to standard, is (+) 2.6%. The value of Column 12 for this trajectory being 1000 yards, the change in range for the whole trajectory, due to the aggregate effect of the several density variations, then is $(-) \frac{2.6}{10} \times 1000 = (-) 260$ yards.

The principal purpose of the foregoing demonstrations of the methods by which air-density weighting factors are established, and by which the latter are used to determine a ballistic density, is to show that Column 12 of the range table is to be used in connection with a ballistic density just as with any other density. In actual practice, the preparation of ballistic densities from actual aloft observations is usually accomplished by aerological parties (see note adjacent to art. 423); or, if aloft observations are not available, the ballistic density is found from Table IV, as already explained in article 424. In either case the ballistic density embodies the air-density weighting factors.

1014. For further convenience in connection with the use of Column 12 of the range table, a table of multipliers for this column is given in Table V, *Range and Ballistic Tables, 1915*. These multipliers are simply the ratios between any given variation in atmospheric density and the tabular variation of 10% on which Column 12 is based. For example, for a 10% variation the multiplier is 1.00, for a 5% variation it is .50, for a 1% variation .10, etc. Signs are appended to these multipliers, to indicate the direction of the change of range that corresponds to the density variation to which the multiplier applies. The arguments for Table V are the surface density factor δ , and the maximum ordinate. For zero height (i.e., zero maximum ordinate), the values given in the table are the multipliers corresponding to the surface density, and they correspond to the relation

$$M = \frac{1 - \delta}{.10}. \quad (1001)$$

For any other height, the multipliers correspond to the ballistic density for a trajectory having a maximum ordinate of that height, and the relation is

$$M = \frac{1 - \delta_b}{.10}. \quad (1002)$$

The following example will illustrate this further.

Given: Surface temperature 52°F., and barometer 30."40.

Find: Compute the multiplier for Column 12 of the range table, (a) for surface density, and (b) for ballistic density corresponding to a maximum ordinate of 10,000 feet.

From Table III we find $\delta = 1.045$, whence, from (1001) we find

$$M = \frac{1 - 1.045}{.10} = (-) .45.$$

Entering Table IV with surface density 1.045 and maximum ordinate 10,000 feet, we find $\delta_s = 1.035$, whence

$$M = \frac{1 - 1.035}{.10} = (-) .35.$$

Both of these results can be obtained directly from Table V.

1015. The multipliers for Column 12 are applicable directly to the values in that column. For example, if the value of the multiplier is $(-) .35$, it means that the change in range for the conditions represented by that multiplier is .35 times the value given in Column 12, and the minus sign indicates the change is a shortening of the range; a plus sign indicates an increase in range.

It is to be noted that if Table V is entered with a ballistic density, the corresponding multiplier must be taken from the line for zero height. This is so because the values in the body of Table V represent the operations both of reducing a surface density factor to the ballistic density, and of deriving from the latter the corresponding multiplier. Therefore if the density factor used in entering this table is already a ballistic density, the only operation remaining to be performed is that indicated by (1002), and since the latter is identical in form with (1001), the required operation is represented in the multiplier for zero height. The practical applications of Column 12, and of the multipliers therefor, will be dealt with further in Chapter 12.

Determination of the multiplier corresponding to a ballistic density

CHANGE IN RANGE DUE TO A GIVEN VARIATION IN WEIGHT OF PROJECTILE (COLUMN 11).

1016. In U. S. Navy range tables, Column 11 gives the change in range due to a small increase or decrease in the weight of the projectile. The variation in weight assumed for this column depends on the caliber of the gun, and is stated in the heading of the column. The heading of Column 11 in the more recent tables states that the values in this column are the changes in range corresponding to a minus variation (i.e., decrease) in the weight of the projectile. This is not intended to imply that the values were derived on the basis of a minus variation only, but merely to indicate that the sign of the tabulated values corresponds to a decrease in weight (the plus sign is to be understood when no sign appears before the tabulated values). A decrease in weight of the projectile generally results in an increase in range, but not necessarily so. It will be observed, for example, that in the case of the 5"3150 f.s. gun, a decrease in weight of projectile *increases* the range for ranges less than

Size and sign of variations assumed for Column 11

8000 yards, and *decreases* the range for ranges greater than 8000 yards. The reason for this will appear presently.

1017. The change in range due to a variation in the weight of the projectile is the resultant of two distinct effects, viz., (1) a change in the initial velocity, and (2) a change in the ballistic coefficient. An increase in the weight of the projectile operates to decrease the range due to a decrease in the initial velocity, and to increase the range due to an increase in the ballistic coefficient. For a decrease in weight the same effects occur with reversed signs. In any case, the two effects oppose each other, and they may even cancel each other (as is seen to be the case, for example, in the 5"3150 f.s. range table at 8000 yards). We shall denote a variation in the weight of the projectile by Δw , the corresponding variations in initial velocity and ballistic coefficient, respectively, by ΔV_w and ΔC_w , and the change in range which is the resultant of both of these effects, by ΔX_w .

The determination of ΔV_w is a problem in interior ballistics; it may be handled either by differentiating the velocity formula with respect to w alone and thus obtaining a relation between ΔV_w and Δw , or by the direct process of solving the velocity formula for any assumed increased and decreased weights of projectile, whence the mean of the values of ΔV_w corresponding to plus and minus values of Δw of any given magnitude may be obtained.* Variations in projectile weight are held within narrow limits in manufacture, and we need concern ourselves, therefore, only with relatively small values of Δw . Changes in initial velocity corresponding to small variations in weight of projectile are given with sufficient accuracy by the formula

$$\Delta V_w = (-)m \times \frac{\Delta w}{w} \times V \quad (1003)$$

in which m is a coefficient whose value varies from about .20 to about .40. The value $m = .36$ has been used for most of our range tables, although for some of the smaller guns smaller values of m have recently been found more satisfactory (for example, $m = .26$ has been adopted for a recent 4" range table). The minus sign before m is accounted for by the fact that a minus velocity variation corresponds to a plus weight variation, and a plus velocity variation to a minus weight variation.

It is evident, from formula (406), that ΔC_w is directly proportional to Δw . Hence we may write

$$\frac{\Delta C_w}{C} = \frac{\Delta w}{w}.$$

Change in
ballistic
coefficient
due to change
in weight of
projectile

A more convenient form of the above relation, for our present purpose, is

$$\Delta C_w = \frac{\Delta w}{w} \times 100 \quad (1004)$$

in which ΔC_w is expressed directly as a percentage variation with respect to C .

1018. Having determined the changes in initial velocity and ballistic coefficient due to a given change in weight of projectile, it is a simple matter to find the corresponding changes in range from Columns 10 and 12, respectively. (It is to be noted that Column 12, although designed to be used primarily in connection

* The necessary formulas, and examples of solution, are given in Section V, Chapter III, *Naval Ordnance*, 1933.

with variations in atmospheric density, may be used also in connection with variations in any other factors contained in C . The sign to be used with values from this column depends, of course, on whether the factor to be dealt with is in the numerator or denominator of the formula for C (406)).

Computation of Column 11 The following example illustrates the determination of values for Column 11 according to the principles that have been outlined above.

Given: In the computation of range-table values for the 16"2600 f.s. gun, for $\phi = 25^\circ$, the values for Columns 10 and 12 have been found to be, respectively, 180 yards and 1292 yards. The standard weight of projectile is 2100 lbs., and the value of the coefficient m is .36.

Find: The change in range due a variation of ± 10 lbs. in weight of projectile.

From (1003) we have

$$\Delta V_w = (-) .36 \times \frac{(\pm) 10}{2100} \times 2600 = (\mp) 4.46 \text{ f.s.}$$

whence from Column 10 we have

$$\Delta X_1 = \frac{(\mp) 4.46}{10} \times 180 = (\mp) 80 \text{ yards.}$$

From (1004) we have

$$\Delta C_w = \frac{(\pm) 10}{2100} \times 100 = (\pm) 0.48\%$$

whence from Column 12 we have

$$\Delta X_2 = \frac{(\pm) .48}{10} \times 1292 = (\pm) 62 \text{ yards.}$$

Combining the two changes, ΔX_1 and ΔX_2 , we have, finally,

$$\Delta X_w = (\mp) 80 (\pm) 62 = (\mp) 18 \text{ yards.}$$

The result we have found is that an increase of 10 lbs. in w decreases the range 18 yards, and a decrease of 10 lbs. in w increases the range 18 yards.*

1019. As a further example, let us take the following.

Given: For the 5"3150 f.s. gun, the following values have been found (all in yards).

Range	Col. 10	Col. 12
1000	6	6
8000	31	352
12000	39	643

The standard weight of projectile is 50 lbs., and the value of the coefficient m is .36.

Find: The change in range due to a variation of ± 1 lb. in weight of projectile (Col. 11) at each of the given ranges.

Proceeding just as in the first example, we have

$$\Delta V_w = (-) .36 \times \frac{(\pm) 1}{50} \times 3150 = (\mp) 22.68 \text{ f.s.}$$

* See note on page 127.

$$\Delta C_w = \frac{(\pm) 1}{50} \times 100 = (\pm) 2\%$$

and these values apply at all ranges. For each of the given ranges the solution is then completed as follows:

For range 1000 yards

$$\Delta X_1 = \frac{(\mp) 22.68}{10} \times 6 = \underline{(\mp) 14 \text{ yards}}$$

$$\Delta X_2 = \frac{(\pm) 2}{10} \times 6 = \underline{(\pm) 1 \text{ yard}}$$

$$\Delta X_w = (\mp) 14 (\pm) 1 = \underline{(\mp) 13 \text{ yards}}$$

For range 8000 yards

$$\Delta X_1 = \frac{(\mp) 22.68}{10} \times 31 = \underline{(\mp) 70 \text{ yards}}$$

$$\Delta X_2 = \frac{(\pm) 2}{10} \times 352 = \underline{(\pm) 70 \text{ yards}}$$

$$w = (\mp) 70 (\pm) 70 = \underline{0}$$

For range 12,000 yards

$$\Delta X_1 = \frac{(\mp) 22.68}{10} \times 39 = \underline{(\mp) 88 \text{ yards}}$$

$$\Delta X_2 = \frac{(\pm) 2}{10} \times 643 = \underline{(\pm) 129 \text{ yards}}$$

$$\Delta X_w = (\mp) 88 (\pm) 129 = \underline{(\pm) 41 \text{ yards}}$$

This example shows that, for the 5"3150 f.s. gun, a decrease in weight of projectile results in an increase of range at ranges less than 8000 yards, and a decrease of range at ranges greater than 8000 yards. This accounts for the change of sign that occurs in Column 11 of this range table at 8000 yards. It is of interest to note that at 8000 yards a change in weight of projectile has practically no effect on the range. In the case of the 16"2600 f.s. gun, for similar reasons, the value in Column 11 reaches a maximum at the range of about 15,000 yards, and beyond that it decreases, although it does not reach the zero value within the limits of the table.

EXERCISES

1. *Given:* The initial velocity, diameter, weight, and coefficient of form of the projectile, and the angle of departure.
Find: The change in range due to a variation of $(-)$ 10 f.s. in initial velocity. (Use A.L.V.F. Tables, and use V to nearest one-tenth of a m.s.).

	Given					Answers
	V (f.s.)	d (in.)	w (lbs.)	i	ϕ	ΔX_V^* (yards)
A	2600	16	2100	.61230	15°	(-)138
B	2600	16	2100	.61200	30	(-)205
C	2600	16	2100	.61140	40	(-)241

* See note on page 127.

2. *Given:* The initial velocity, diameter, weight, and coefficient of form of the projectile, and the angle of departure.

Find: The change in range due to a variation of (\pm) 10% in atmospheric density. (Use A.L.V.F. Tables.)

	Given					Answers
	V (f.s.)	d (in.)	w (lbs.)	i	ϕ	ΔX_C^* (yards)
A	2600	16	2100	.61230	15°	(+) 778
B	2600	16	2100	.61200	30	(+) 1498
C	2600	16	2100	.61140	40	(+) 1752

* See note on page 127.

3. *Given:* The range, initial velocity, weight of projectile, coefficient m , and the values from Columns 10 and 12 of the range table (16"2600 f.s. gun).

Find: The change in range due to the stated variation in weight of projectile, for the given range.

	Given							Answers
	X (yards)	V (f.s.)	w (lbs.)	Δw (lbs.)	m	Col. 10 (yards)	Col. 12 (yards)	ΔX_w^* (yards)
A	3000	2600	2100	(\pm)10	.36	22	16	(\pm) 9
B	8000	2600	2100	(\pm)10	.36	55	107	(\pm) 20
C	20000	2600	2100	(\pm)10	.36	115	594	(\pm) 22

* See note on page 127.

CHAPTER 11

THE DETERMINATION OF THE EFFECTS OF WIND AND MOTION OF GUN ON THE TRAJECTORY, AND OF MOTION OF TARGET ON THE POINT OF FALL WITH RESPECT TO THE TARGET (RANGE-TABLE COLUMNS 13-18).

New Symbols Introduced

W_x, G_x, T_x	Components of wind, gun motion, and target motion, respectively, in the line of fire.
W_z, G_z, T_z	Components of wind, gun motion, and target motion, respectively, perpendicular to the line of fire.
$\Delta\phi_w, \Delta\phi_g$	Apparent changes in ϕ due to wind and gun motion, respectively, in the line of fire.
$\Delta V_w, \Delta V_g$	Apparent changes in V due to wind and gun motion, respectively, in the line of fire.
ΔX_ϕ	The change in range corresponding to a change in ϕ (as for example $\Delta\phi_w$ or $\Delta\phi_g$).
$\Delta X_w, \Delta X_g, \Delta X_T$	Changes in range due to components of wind, gun motion, and target motion, respectively, in the line of fire.
D_w, D_g, D_T	Lateral deviations due to components of wind, gun motion, and target motion, respectively, perpendicular to the line of fire.

1101. Columns 13-18 of the range table deal with the effects of wind, motion of gun, and motion of target, all of which are assumed to occur in a horizontal plane through the origin or in a plane parallel thereto. Columns 13, 14, and 15, deal, respectively, with the effects of components of wind, motion of gun, and motion of target occurring wholly *in* the line of fire, and Columns 16, 17, and 18, respectively, with the effects of such components occurring wholly *perpendicular* to the line of fire (all in the horizontal plane).

It was formerly the practice to base Columns 13-18 on 12-knot components, and a few range tables based on this practice are still in use. The present practice is to base these columns on 10-knot components. The size of the component on which these columns are based is always stated in the headings of the columns. No signs appear in these columns, but the proper sign can always be inferred from the sign of the component itself; this matter will be dealt with further in Chapter 12. All of the examples and exercises given in this text will be based on 10-knot components, and on the relation 1 sea mile = 6080 feet, whence 1 knot = 1.689 foot-seconds.

It will be convenient to take up first the effects of target motion, Columns 15 and 18, since the values for these columns can be used to advantage in the computation of the other columns.

CHANGE IN RANGE DUE TO A GIVEN COMPONENT OF TARGET MOTION IN THE LINE OF FIRE (COLUMN 15).

1102. Motion of the target in no way affects the trajectory itself, but nevertheless it affects the position of the point of fall with respect to the target, and

hence its result is the equivalent of a disturbance of the trajectory itself. We evidently must lay the gun in elevation, not for the range to the target at the instant of firing, but for this range plus or minus the change in range that ensues from the target's motion during the time of flight.

We shall denote by T_x the component of target motion in the line of fire, and by ΔX_T the change in range resulting therefrom. The latter amounts to the product of the target motion and the time of flight, or

$$\Delta X_T = T_x \times T. \quad (1101)$$

Strictly speaking, the value of T used in (1101) should be the time of flight corresponding to the corrected range, $X \pm \Delta X_T$, rather than the time of flight for the assumed range X (i.e., the range to the target at the instant of firing). If this distinction is to be made in the value of T , then (1101) must be solved by successive approximations, the first approximation being based on the value of T which corresponds to the assumed range X ; further approximations can then be based on values of T corresponding to the corrected ranges $X \pm \Delta X_T$. The values given in Column 15 are the first approximations of ΔX_T , and hence are slightly too great for target motion toward the gun and slightly too small for target motion away from the gun. If necessary, closer approximations can be derived from the tabular values, without resorting to additional computations, as will be illustrated presently.

1103. The computation of values for Column 15 is illustrated in the following example.

Given: The 16" 2600 f.s. gun, range 31,200 yards, for which the time of flight equals 57.79 seconds.
Computation of Column 15 *Find:* The change in range due to a 10-knot component of target motion in the line of fire (Column 15).

Applying formula (1101), we have

$T_x = 10$ knots	log 1.00000
1.689 (art. 1101)	log 0.22763
$T = 57.79$ seconds	log 1.76185
$\Delta X_T = 976.08$ feet	log 2.98948
<u> = 325 yards</u>	

The result thus obtained means that if the sight-bar range of 31,200 yards will cause a hit on a motionless target, it will cause a miss 325 yards short of a target moving at ten knots directly away from the gun, and a miss 325 yards beyond a target moving at ten knots directly toward the gun. However, if we change the sight-bar range to $31,200 + 325 = 31,525$ yards for the target moving away, the point of fall will still occur slightly short of the target, due to the increase in time of flight for the increased sight-bar range. From Column 15 we find that during the time of flight corresponding to the range 31,525 yards, the motion of the target is 330 yards, and hence a more accurate determination of the sight-bar range required to intercept the target is $31,200 + 330 = 31,530$ yards. Further approximations will not change the latter result. Proceeding similarly with the case of a target moving at 10 knots directly toward the gun, we find as a first approximation of the corrected sight-bar range $31,200 - 325 = 30,875$ yards, and as a second (and final) approximation $31,200 - 320 = 30,880$ yards. In either of the above cases the difference between ΔX_T as found directly from the range table and as found by successive approxi-

Accuracy of Column 15

mations, is of no practical consequence, especially in comparison with the errors that are likely to result from inaccurate knowledge of the target speed itself. This remains true even for the greatest target speeds that are likely to be encountered in surface fire. It is the usual practice, therefore, to use the values from Column 15 as they are given in the range table (i.e., without resorting to the further approximations outlined above).

LATERAL DEVIATION DUE TO A GIVEN COMPONENT OF TARGET MOTION PERPENDICULAR TO THE LINE OF FIRE (COLUMN 18).

1104. The computation of values for Column 18 is identical with that for Column 15. We shall denote by T_z the component of target motion perpendicular to the line of fire, and by D_T the lateral deviation resulting therefrom. Then we have

$$D_T = T_z \times T \quad (1102)$$

in which T is the time of flight corresponding to the given range. Since the range is not appreciably affected by D_T , the time of flight also is not appreciably affected, and a direct solution of (1102) gives as great a degree of accuracy as is required for any purpose. The computation for Column 18 is illustrated in the following example.

Given: The 16" 2600 f.s. gun, range 31,200 yards, for which the time of flight equals 57.79 seconds.
Computation of Column 18 *Find:* The lateral deviation due to a 10-knot component of target motion perpendicular to the line of fire (Column 18).

Applying formula (1102), we have

$T_z = 10$ knots.	log 1.00000
1.689 (art. 1101)	log 0.22763
$T = 57.79$	log 1.76185
$D_T = 976.08$ feet	log 2.98948
= 325 yards	

The result thus obtained means that, in this case, the gun must be directed 325 yards to the right if the target motion is to the right, and 325 yards to the left if the target motion is to the left, in order that the point of fall may occur on the moving target. The method by which the gun is offset to counteract lateral deviations will be explained in Chapter 12.

CHANGE IN RANGE DUE TO A GIVEN WIND COMPONENT IN THE LINE OF FIRE (COLUMN 13).*

1105. Although it is well known that the wind may at times have vertical components of appreciable magnitude, practical methods of measuring the wind are limited to the determination of its horizontal components, and provisions for the practical determination of the effects of wind on the trajectory are accordingly limited to horizontal winds. Lack of knowledge of vertical wind components is, however, not the only limitation involved in our determination of the effects

* See also Appendix D.

Limitations involved in correcting for effects of wind of the wind, for even in the measurement of the horizontal wind we are limited to a relatively low order of approximation, because of the practical difficulties involved in obtaining aloft observations that are immediately applicable to the time of firing. Moreover, actual aloft observations may often be lacking altogether, in which case the determination of aloft winds rests entirely on estimate. It is to be appreciated, therefore, that the practical determination of the effects of wind on the trajectory may often be largely a matter of speculation.

The range-table columns which deal with the effects of wind on the trajectory are based on a horizontal wind which is considered to be of uniform magnitude and direction throughout the trajectory. Column 13 is based on a component of such wind measured *in* the line of fire, and Column 16 on a component measured *perpendicularly* to the line of fire. We shall refer to these components, respectively, as *range wind* and *cross wind*.

General principles governing determination of effects of wind 1106. In analyzing the effect of wind on the flight of the projectile, it is convenient to consider separately the motion of the projectile with respect to the moving air, and the motion of the air itself. If the air is in motion away from the gun, we may consider, then, that the initial velocity of the projectile with respect to this moving air is less than it would be with respect to still air, and that the projectile consequently travels a shorter distance through the moving air itself than it would travel through the still air itself. However, while the projectile in this case travels a shorter distance with respect to the air, the latter itself moves bodily forward, and the travel of the projectile with respect to the ground is the resultant of its travel through the air and of the latter's motion over the ground. Similar reasoning applies to the case of air moving toward the gun; in this case the projectile travels a greater distance with respect to the air, but the latter itself moves backward. The loss or gain of distance with respect to the moving air is always less than the accompanying gain or loss of distance due to the motion of the air over the ground, whence it follows, as is to be expected, that a wind blowing away from the gun increases the range and a wind blowing toward the gun decreases the range.

Let us denote by W_x the range wind (positive for wind blowing away from the gun, and negative for wind blowing toward the gun), and by ΔX_w the change in range resulting therefrom. Then if X_1 represents the range with respect to the moving air, as compared to the range X in still air, we may write

$$\Delta X_w = X_1 \pm W_x T - X. \quad (1103)$$

In the above formula, the range over the ground, under the condition of a wind W_x , is represented by $X_1 \pm W_x T$; for X_1 represents the range with respect to the moving air, and $W_x T$ the distance through which the air itself moves. Strictly speaking, T should be the time of flight for the range X_1 , but no material error is occasioned by taking it as the time of flight corresponding to the range X . (The situation here is comparable to that already discussed in article 1102.)

1107. The above analysis of the effect of a range wind is illustrated graphi-

cally in Figure 22, which represents the case of a wind blowing away from the gun. In the upper diagram, OSH represents the trajectory that results from given values of ϕ and V when no wind is blowing, the range in this case being $X = OH$; $OS'H'$ represents the trajectory that results from the same ϕ and V when a wind

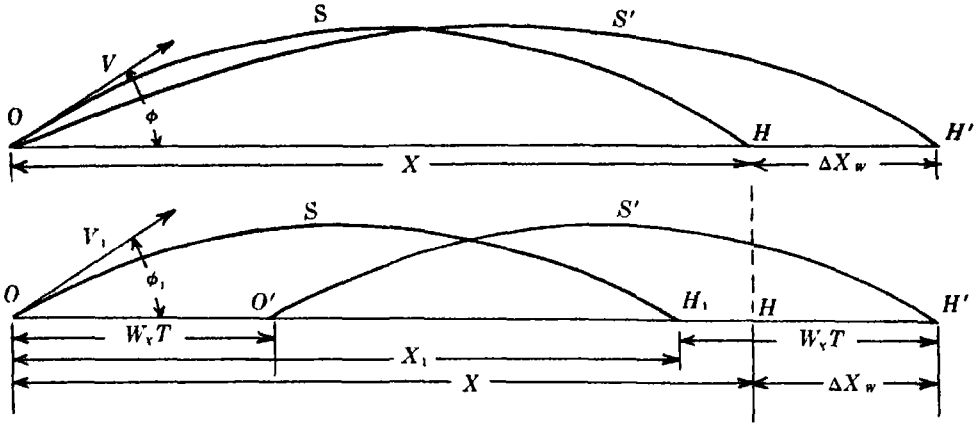


FIGURE 22

Graphical
representation
of effects of
range wind

W_x is blowing away from the gun, the range now being increased by $\Delta X_w = HH'$. In the lower diagram, the effect of the wind in increasing the range is analyzed into the two effects already discussed; OSH_1 is the trajectory with respect to the moving air, and $O'S'H'$ is the same trajectory translated forward through the distance $W_x T$. It is to be understood that the same *actual* ϕ and V are considered to apply in the lower diagram as in the upper diagram, although the trajectory OSH_1 of the lower diagram, and its corresponding range $X_1 = OH_1$, are assumed to result from an *apparent* angle of departure ϕ_1 and initial velocity V_1 . The values ϕ_1 , V_1 , and X_1 of the lower diagram are all to be understood as being measured in relation to the moving air. Figure 22 shows that $\Delta X_w = HH'$ is the resultant of a decrease HH_1 in range which the projectile suffers with respect to the moving air, and an increase $W_x T = H_1 H'$ in range which it gains due to the bodily movement of the air itself.

1108. In order to solve formula (1103), we must determine the value of X_1 , i.e., the range of the trajectory with respect to the moving air. This can be done by finding the values of the angle of departure and initial velocity with respect to the moving air; we shall denote these by ϕ_1 and V_1 . In Figure 23 (a) the actual initial velocity is represented by the vector $V = OA$, making with the horizontal the angle AOH which is equal to the actual angle of departure ϕ ; in other words, OA represents vectorially the initial motion imparted to the projectile with respect to the ground. In the same figure, AB represents vectorially a horizontal wind component W_x blowing away from the gun. OB then represents vectorially the initial motion of the projectile with respect to the moving air, and $V_1 = OB$ and $\phi_1 = BOH$ are the initial velocity and angle of departure with respect to the moving air. Figure 23 (b) similarly represents the case of a wind blowing toward

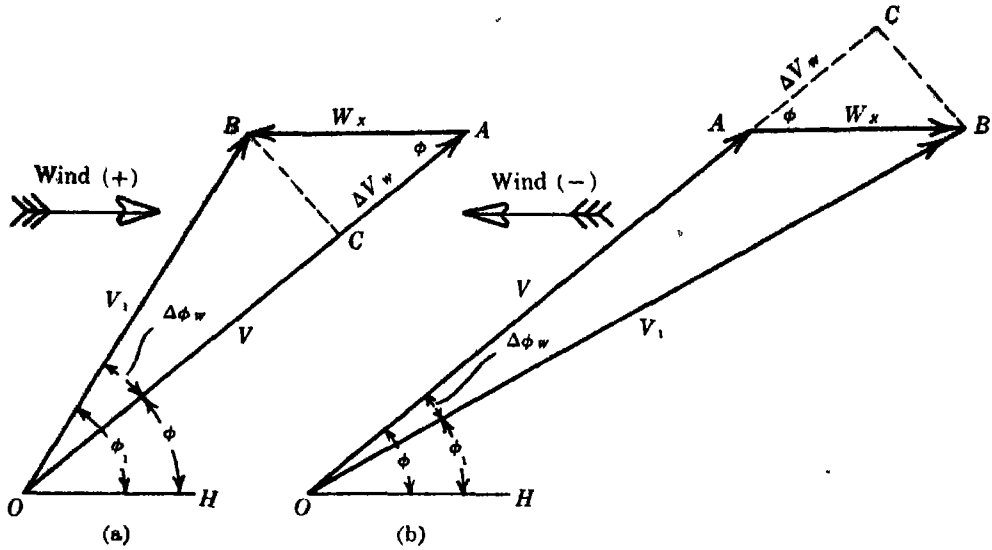


FIGURE 23

the gun. (The direction of the vector AB is correctly drawn in each case to show that OB is the resultant of OA and AB .)

It is seen that with a positive wind (Figure 23 (a)) there is an apparent increase in angle of departure and an apparent decrease in initial velocity, with respect to the moving air, while with a negative wind (Figure 23 (b)) there is an apparent decrease in angle of departure and an apparent increase in initial velocity, with respect to the moving air. We shall denote these apparent changes in angle of departure and initial velocity by $\Delta\phi_w$ and ΔV_w . If we drop a perpendicular from B to C in either of the above figures we have, in each case, OC practically equal to OB (since the angle AOB between the vectors OA and OB is always very small). Then AC , in each case, is practically equal to the difference in length between the two vectors OA and OB , and hence is practically equal to ΔV_w , which represents the difference between V and V_1 . From the construction of the figure we have, in either case, $\Delta V_w = W_x \cos \phi$ (approximately). Also, in either case,

$$\sin \Delta\phi_w = \frac{BC}{OB}. \text{ But } BC = W_x \sin \phi, \text{ and } OB = V_1, \text{ whence } \sin \Delta\phi_w = \frac{W_x \sin \phi}{V_1}.$$

In any practical case, however, the difference between V_1 and V , although sufficiently great to affect the range appreciably, is not great enough to affect the value of $\Delta\phi_w$ appreciably. Therefore no material error is occasioned by using V in place of V_1 to find $\Delta\phi_w$ for both positive and negative winds.

1109. The apparent changes in V and ϕ with respect to the moving air can therefore be found with sufficient approximation from the expressions

$$\Delta V_w = W_x \cos \phi \quad (1104)$$

$$\sin \Delta\phi_w = \frac{W_x \sin \phi}{V} \quad (1105)$$

and the apparent initial velocity and angle of departure resulting from these changes can then be found from the expressions

$$V_1 = V \pm \Delta V_w \quad (1106)$$

$$\phi_1 = \phi \pm \Delta\phi_w. \quad (1107)$$

With regard to signs, it is to be noted that the sign of $\Delta\phi_w$ must always be contrary to that of ΔV_w . In the case of a positive wind, $\Delta\phi_w$ is positive and ΔV_w negative; in the case of a negative wind, $\Delta\phi_w$ is negative and ΔV_w positive. Reference to Figures 23 (a) and (b) will make this clear.

The range corresponding to ϕ_1 and V_1 is the range with respect to the moving air, or the value X_1 appearing in formula (1103). Since $\Delta\phi_w$ and ΔV_w are always small, we can determine their effects on the range readily from data already computed for the range table, as follows. Let us denote by ΔX_ϕ the change in range corresponding to the change $\Delta\phi_w$ in angle of departure, and by ΔX_v the change in range corresponding to the change ΔV_w in initial velocity. The value of ΔX_ϕ can then be found from Column 2 (b) of the range table,* which gives the value of $\Delta\phi$ corresponding to $\Delta X = 100$ yards. The value of ΔX_v can, of course, be found from Column 10. In accordance with the rule for signs given in the preceding paragraph, we can then write,

(a) for a positive wind (+ W_x)

$$X_1 = X + \Delta X_\phi - \Delta X_v$$

whence from (1103)

$$\begin{aligned}\Delta X_w &= X + \Delta X_\phi - \Delta X_v + W_x T - X \\ &= + (W_x T + \Delta X_\phi - \Delta X_v)\end{aligned}$$

(b) for a negative wind (- W_x)

$$X_1 = X - \Delta X_\phi + \Delta X_v$$

whence from (1103)

$$\begin{aligned}\Delta X_w &= X - \Delta X_\phi + \Delta X_v - W_x T - X \\ &= - (W_x T + \Delta X_\phi - \Delta X_v).\end{aligned}$$

It follows from the above that the change in range due to a wind component W_x in the line fire, can be found from the formula

Formula for
determination
of effect of
range wind

$$\Delta X_w = W_x T + \Delta X_\phi - \Delta X_v \quad (1108)$$

in which ΔX_w takes the same sign as the wind component itself, and in which T is the time of flight corresponding to the given range X , ΔX_ϕ is the change in range due to a change $\Delta\phi_w$ in angle of departure as found from (1105), and ΔX_v is the change in range due to a change ΔV_w in initial velocity as found from (1104). A further simplification of the above results from the fact that the term $W_x T$ is identical in form with the formula $T_x T$ (1101) from which Column 15 of the range table has already been computed. For a 10-knot component of wind ($W_x = 10$ knots), the value of $W_x T$ can therefore be taken directly from Column 15, which is also based on a 10-knot component. Formula (1108) can then be written finally

$$\Delta X_w = (\text{Col. 15}) + \Delta X_\phi - \Delta X_v. \quad (1109)$$

1110. The solution of (1109) is illustrated in the following example.

Computation
of Column 13

Given: The 16"2600 f.s. gun, $\phi = 25^\circ$, and the following range-table values corresponding to this angle of departure: Col. 2 (b) = 8'.7, Col. 10 = 180 yards, Col. 15 = 325 yards.

Find: The change in range due to a wind component of 10 knots in the line of fire (Column 13).

* Column 2 (b) does not appear in the earlier range tables, but the required data can always be found readily from Column 2.

Solving (1104) and (1105), we have

$$\begin{array}{llll}
 W_x = 10 \text{ knots} & \dots\dots\dots \log 1.00000 & \dots\dots\dots \log 1.00000 \\
 1.689 \text{ (art. 1101)} & \dots\dots\dots \log .22763 & \dots\dots\dots \log .22763 \\
 \phi = 25^\circ & \dots\dots\dots \lrcos 9.95728 - 10 & \dots\dots\dots \lrcos 9.62595 - 10 \\
 \Delta V_w = 15.3 \text{ f.s.} & \dots\dots\dots \log 1.18491 & & \\
 V = 2600 \text{ f.s.} & \dots\dots\dots \text{colog } 6.58503 - 10 & & \\
 \Delta \phi_w = 9'.5 & \dots\dots\dots \lrcos 7.43861 - 10 & &
 \end{array}$$

From Column 2 (b) we have

$$\Delta X_\phi = \frac{9.5}{8.7} \times 100 = \underline{109 \text{ yards}}$$

and from Column 10

$$\Delta X_V = \frac{15.3}{10} \times 180 = \underline{275 \text{ yards}}$$

whence from formula (1109)

$$\Delta X_w = 325 + 109 - 275 = \underline{159 \text{ yards}.^*}$$

LATERAL DEVIATION DUE TO A GIVEN WIND COMPONENT PERPENDICULAR TO THE LINE OF FIRE (COLUMN 16).†

1111. In analyzing the effect of a cross wind on the flight of the projectile, we shall again consider separately the motion of the projectile with respect to the moving air, and the motion of the air itself. If the air is in motion *from* the left, we may consider, then, that the projectile has an apparent motion *to* the left with respect to the air.‡ However, while the projectile is apparently moving to the left with respect to the air, the latter is itself moving bodily to the right. Similar reasoning applies to the case of air moving from the right. The projectile's lateral displacement to the left or right with respect to the moving air, is always less than the accompanying lateral displacement to the right or left of the air itself,

* By a similar treatment, the effect of a vertical wind component W_Y is found to be the resultant of a bodily raising or lowering of the entire trajectory by the amount $W_Y T$, and of an apparent change in the trajectory with respect to the moving air. The change in range corresponding to the change in height $W_Y T$ can be found from Column 19; it is positive for a wind blowing upward, and negative for a wind blowing downward. From a vector analysis similar to that given in Figures 23 (a) and (b), it can be shown that for a vertical wind the ap-

parent changes in ϕ and V with respect to the moving air, are $\sin \Delta \phi_w = \frac{W_Y \cos \phi}{V}$ and

$\Delta V_w = W_Y \sin \phi$, and that they are of like sign, both negative for a wind blowing upward, and both positive for a wind blowing downward. For a 10-knot vertical wind, in the case of the above example, the resultant change in range due to the operation of the three effects is $443 - 232 - 128 = 83$ yards. An examination of the effect of vertical wind at other angles of departure (16"2600 f.s. gun) shows that the effect has its greatest value, about 120 yards for a 10-knot component, at about $\phi = 15^\circ$. For greater or lesser values of ϕ it decreases, becoming about 100 yards at $\phi = 5^\circ$, and about 40 yards at $\phi = 40^\circ$. Although these values undoubtedly are only rough approximations, they serve to indicate that the degree of error occasioned by ignorance of vertical wind components is of less consequence than might be supposed, in long-range fire. On the other hand, it is worthy of note that at short ranges the effect of vertical wind far exceeds that of a horizontal wind of equal magnitude.

See also Appendix D.

† Right and left are to be considered as signifying directions referred to the direction of

whence it follows, as is to be expected, that a wind blowing *from the left* causes a lateral deviation of the projectile *to the right*, and a wind blowing *from the right* a lateral deviation of the projectile *to the left*.

Let us denote by W_z the cross wind (positive for wind blowing *from* the left, and negative for wind blowing *from* the right), and by D_w the lateral deviation resulting therefrom. Then if D_1 represents the lateral deviation with respect to the moving air, we may write

$$D_W = W_z T - D_1 \quad (1110)$$

which states simply that the lateral deviation with respect to the ground is the resultant of the lateral deviation with respect to the moving air and of the lateral displacement of the air itself during the time of flight.

Graphical representation of effects of cross wind

1112. The above analysis of the effect of a cross wind is illustrated graphically in Figure 24 (a), which represents the case of a wind blowing from the left (i.e., a positive cross wind). Figure 24 (a) is a plan view in which OH , OH_1 and OH' represent the traces, on the horizontal plane, of the trajectories to which we shall refer in the following discussion. OH represents the trajectory unaffected by wind, OH_1 the trajectory affected by the wind W_z and as described with respect to the moving air, and OH' the trajectory affected by the same wind W_z but as described with

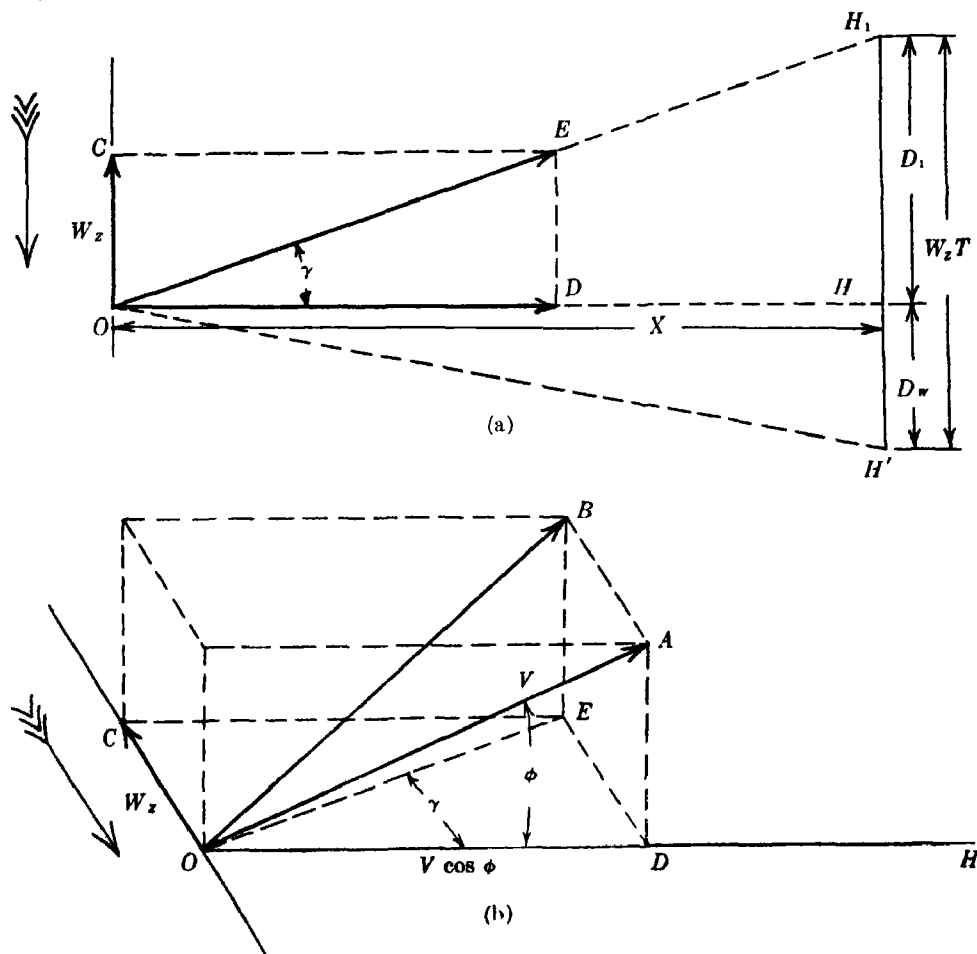


FIGURE 24

respect to the ground. The range is represented by $X = OH$. No distinction as to range need be made among OH , OH_1 , and OH' ; the differences among them, although exaggerated in the figure, are actually insignificant in any practical case. The lateral deviation with respect to the moving air is represented by $D_1 = HH_1$, or $D_1 = X \tan \gamma$ in which γ is the angle DOE . The lateral displacement of the air during the time of flight is represented by $H_1H' = W_zT$. The resultant lateral deviation with respect to the ground is then $HH' = D_w$, and we have

$$D_w = W_zT - X \tan \gamma. \tag{1111}$$

The angle γ which is required for the solution of (1111) can be found from the vector analysis illustrated in Figure 24 (b), which is an elevation view, in perspective, corresponding to the plan view of Figure 24 (a). In Figure 24 (b) the actual initial velocity is represented by the vector $V = OA$, making with the horizontal the angle $AOD = \phi$, and the cross-wind is represented by the vector $W_z = OC$. The resultant of these two vectors is the vector OB , which represents the initial motion of the projectile with respect to the moving air. The vector OB represents a slight increase in initial velocity and decrease in angle of departure, as compared to OA , but these differences are too small to affect the range appreciably. The essential difference between the two vectors is the lateral angular displacement of OB with respect to OA , which is measured in the horizontal plane, as illustrated, by the angle $DOE = \gamma$, OD and OE being the traces, on the horizontal plane, of the vectors OA and OB , respectively. From the construction it is evident that

$$\tan \gamma = \frac{W_z}{V \cos \phi}.$$

Substituting this value in (1111), we have

$$D_w = W_zT - \frac{XW_z}{V \cos \phi} \tag{1112}$$

Formula for
determination
of effect of
cross wind

or

$$D_w = W_z \left(T - \frac{X}{V \cos \phi} \right). \tag{1113}$$

1113. The solution of (1113) is illustrated in the following example.

Computation
of Column 16

Given: The 16"2600 f.s. gun, $\phi = 25^\circ$, range 31,200 yards, time of flight 57.79 seconds.

Find: The lateral deviation due to a 10-knot component of wind perpendicular to the line of fire (Column 16).

$X = 93,600$ feet.....	log 4.97128
$V = 2600$ f.s.	colog 6.58503 - 10
$\phi = 25^\circ$	lsec 0.04272
(-) 39.72.....	log 1.59903
$T = 57.79$	
18.07.....	log 1.25696
$W_z = 10$ knots.....	log 1.00000
1.689 (art. 1101).....	log 0.22763
$D_w = 305.20$ feet.....	log 2.48459
<u>= 102 yards</u>	

THE DETERMINATION OF THE EFFECTS OF WINDS WHICH ARE NOT UNIFORM THROUGHOUT THE TRAJECTORY.

1114. Except in the case of small angles of departure, it is to be expected that both the velocity and the direction of the wind will vary materially within the limits of the trajectory, for the velocity and direction of the wind ordinarily vary materially with the altitude. The problem of predicting the effects of such varying winds is quite similar to that of predicting the effects of the different density variations that occur within the limits of a trajectory, and it is handled in

quite the same manner as the latter. The principles which govern the establishment of air-density weighting factors, as outlined in articles 1010-1011, are applied also in the establishment of wind weighting factors; the latter express the ratios of the effects of winds which are confined within given zones of altitude, to the effects of equal winds which operate uniformly throughout the entire trajectory. By applying the wind weighting factors for the several zones of a trajectory to the winds measured in these zones, a

weighted mean wind, or *ballistic wind* is found. The ballistic wind is then evidently a fictitious uniform wind which is the equivalent of the several winds that actually exist within the limits of the trajectory. Wind weighting factors are not the same for range wind as for cross wind, and hence it follows that a theoretically correct determination of the range-wind component depends upon a ballistic wind based on range-wind weighting factors, and of the cross-wind component upon a ballistic wind based on cross-wind weighting factors; in other words, it may be said that two different ballistic winds exist for the same trajectory, one of which may be used only for finding the range wind and the other only for the cross wind.

However, the measurement of aloft winds is subject to practical limitations of about the same order as apply in the measurement of aloft air densities. It is logical practice, therefore, to avoid great elaboration in the establishment of wind

weighting factors, as has already been done in the case of air-density weighting factors (art. 1011). A comprehensive analysis of wind weighting factors, made at the Aberdeen Proving Ground, has resulted in the establishment of a single formula, applicable either to range wind or to cross wind, from which sufficiently approximate weighting factors for any surface trajectory may be found.* The following table is based on this formula, and gives the wind weighting factors for the same zones that have already been used in connection with air-density weighting factors. In arrangement and use, this table is quite similar to the table of air-density weighting factors that has already been given in article 1012.

* The formula is

$$1 - p = 0.74 (1 - k)^{1/2} + 0.26 (1 - k)^2$$

in which p denotes the weighting factor, and k denotes the ratio y/y_0 , to which the weighting factor pertains (see also note adjacent to article 1012). Details entering into the derivation of wind weighting factors are given in Chapter XV, *A Course in Exterior Ballistics* (War Department Document No. 1051, December, 1920) by R. S. Hoar, and in *Computation of Firing Tables for the U. S. Army* (Aberdeen Proving Ground) by H. P. Hitchcock.

The U. S. Army now uses two sets of wind weighting factors, one for ordinary terrestrial fire (this being the same as the one given here) and one for antiaircraft and other high-angle fire. These weighting factors are given in Tables I and III, *Technical Regulations No 1236-1* (U. S. War Department, June, 1934).

Wind Weighting Factors

Maximum ordinate (feet)	Zones (feet)										
	0-600	600-1,500	1,500-3,000	3,000-4,500	4,500-6,000	6,000-9,000	9,000-12,000	12,000-15,000	15,000-18,000	18,000-24,000	24,000-30,000
600	1.00
1,500	.33	.67
3,000	.17	.24	.59
4,500	.11	.17	.26	.46
6,000	.09	.12	.20	.20	.39
9,000	.06	.08	.14	.13	.13	.46
12,000	.04	.07	.10	.10	.10	.20	.39
15,000	.04	.05	.08	.08	.08	.16	.16	.35
18,000	.03	.04	.07	.07	.07	.13	.13	.14	.32
24,000	.02	.04	.05	.05	.05	.10	.10	.10	.10	.39
30,000	.02	.03	.04	.04	.04	.08	.08	.08	.08	.16	.35

Determination of ballistic wind 1115. The following example illustrates a convenient method of determining the ballistic wind from aloft wind observations, by means of the above table.

Given: Aloft winds have been measured as follows (directions are the true compass directions *from* which the wind is blowing).

Zone (feet)	Velocity (knots)	Direction
0-600	15	270°
600-1,500	12	250°
1,500-3,000	15	260°
3,000-4,500	17	270°
4,500-6,000	22	290°
6,000-9,000	26	320°

Find: (a) The ballistic wind for a trajectory whose maximum ordinate is 9,000 feet. (b) The range wind and cross wind corresponding to this ballistic wind and to a line of fire whose true compass direction is 100°.

We shall first weight the winds for the several zones as follows:

Zone	Observed velocity	Weighting factor	Weighted velocity
0-600	15	.06	0.9
600-1,500	12	.08	1.0
1,500-3,000	15	.14	2.1
3,000-4,500	17	.13	2.2
4,500-6,000	22	.13	2.9
6,000-9,000	26	.46	12.0

The weighted velocities for the several zones may now be plotted as vectors, using the observed directions of the winds for their respective zones. A mooring-board diagram, or any other form of polar plotting sheet, is convenient for this purpose. Figure 25 illustrates the method of plotting the vectors. The closing vector, *OW*, of the polygon represents the ballistic wind both as to velocity and as to direction. (Since it is customary in stating wind directions to give the direction *from* which the wind is blowing, care must be taken to plot the wind vectors correctly, i.e., with their arrow-heads pointing in the direction *toward* which the wind is blowing.)

The ballistic wind, as found above, is practically 19 knots *from* 300° true. The direction of the line of fire being 100° true, we find that the wind vector

makes an angle of 20° with the line of fire. The component in the line of fire (range wind) is therefore $19 \cos 20^\circ = 17.9$ knots, and the component perpendicular to the line of fire (cross wind) is $19 \sin 20^\circ = 6.5$ knots; both of the components are positive. The same result can be obtained graphically, by dropping a perpendicular from the extremity of the ballistic-wind vector OW to the line of fire, as shown in the figure; OF is the range wind, and FW the cross wind.

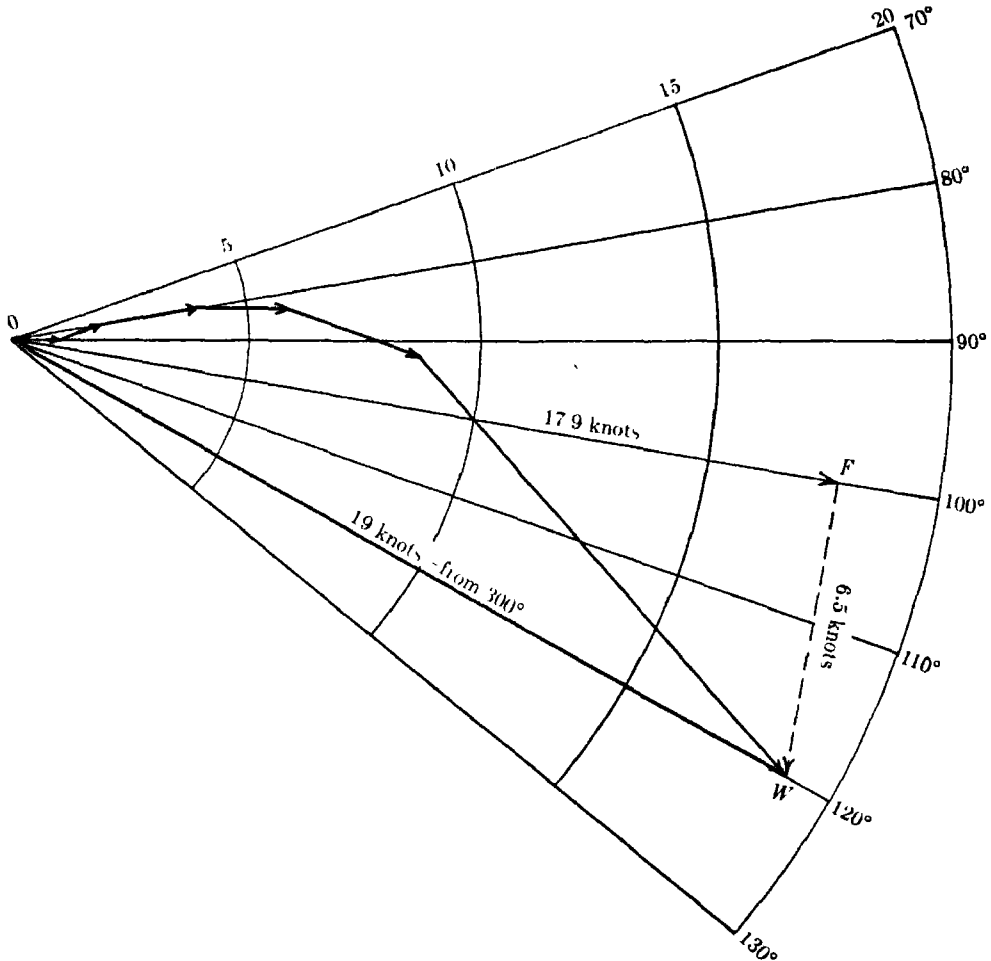


FIGURE 25

1116. It is evidently also possible to resolve the wind for each zone into its range component and cross component, and to weight these components and thus obtain a ballistic range component and ballistic cross component. This method, however, has the disadvantage of delaying the solution for the ballistic wind until the direction of the line of fire is known, and in naval gunnery this is an important disadvantage.* By means of the method illustrated above, the ballistic wind can be determined immediately after aloft soundings have been taken, and all that remains to be done when the direction of fire becomes known is to resolve this single wind into its two components. Special facilities for accomplishing this

* It is doubtful whether any real advantage is to be gained, in any case, by the elaboration of using different weighting factors for the determination of ballistic range wind and ballistic cross wind.

rapidly are provided in plotting rooms or control stations; in the latest equipment, the wind can be applied directly to the range keeper, which automatically resolves it into the required components and generates the corresponding corrections according to Columns 13 and 16 of the range table.

The principal point to be grasped at this time is that the ballistic wind is used with Columns 13 and 16 just as any other wind would be used, and that it ordinarily gives a much more nearly correct result than can be obtained with the surface wind alone. For example, let us suppose that a battery of 16" 2600 f.s. guns is to be fired under the conditions assumed for the problem solved in the foregoing article; the maximum ordinate of 9,000 feet corresponds to a horizontal range of 27,200 yards for this battery. At this range the wind component in the line of fire (17.9 knots) increases the range by 213 yards (Col. 13), and the component perpendicular to the line of fire (6.5 knots) causes a lateral deviation of 50 yards to the *right* (Col. 16). Had we used simply the surface wind (15 knots from 270°), we would have found a component of 14.8 knots in the line of fire and a corresponding increase in range of 176 yards, and a component of 2.6 knots (minus) perpendicular to the line of fire and a corresponding lateral deviation of 20 yards to the *left*. Thus the use of the surface wind in this case would have made matters worse, in deflection, than if the wind had been ignored altogether. The example cited here by no means represents an exaggerated situation. It is not uncommon for the surface wind to differ materially from aloft winds, and for corrections based only on the surface wind to be opposite in sign to those based on the ballistic wind.

Errors
resulting
from use
of surface
wind alone

CHANGE IN RANGE DUE TO A GIVEN COMPONENT OF GUN MOTION IN THE LINE OF FIRE (COLUMN 14).

1117. Let us denote by G_x the component of gun motion in the line of fire (positive when in the direction of fire and negative when contrary thereto), and

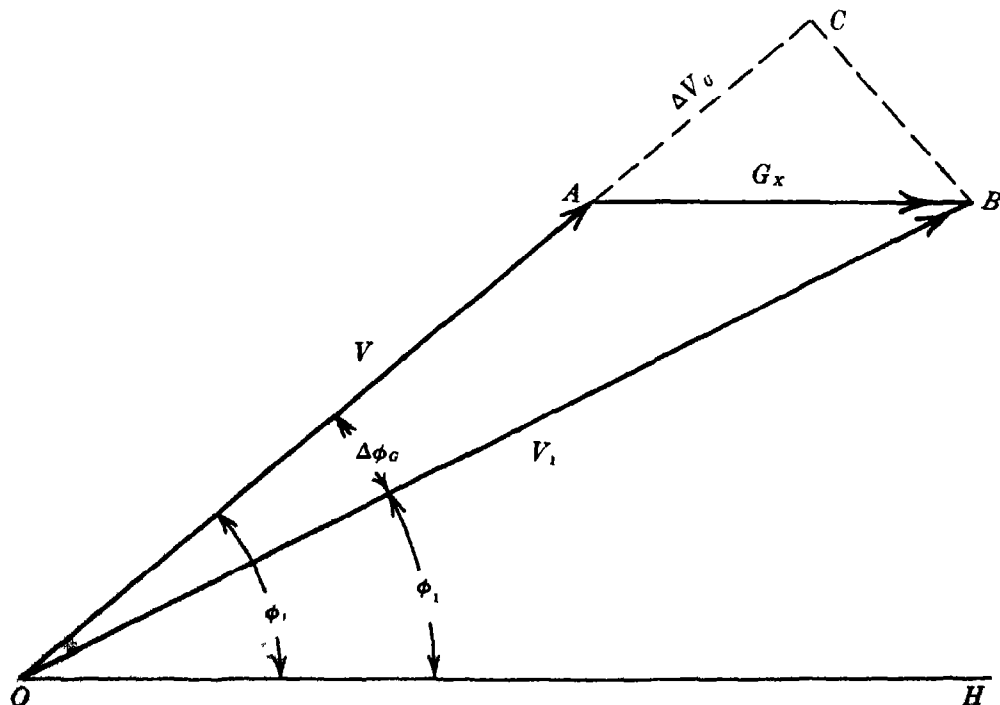


FIGURE 26

by ΔX_G the change in range resulting from such motion. A vector analysis of the effect of gun motion on the initial motion imparted to the projectile shows that a positive component of gun motion causes an increase in initial velocity but also an apparent decrease in angle of departure, while a negative component causes a decrease in initial velocity which is accompanied by an apparent increase in angle of departure. The situation for a positive component of gun motion is illustrated in Figure 26. The construction of this figure is similar to that of 23 (b), and by the same process of reasoning that has already been applied in connection with the latter (arts. 1108-1109), we find that the changes in initial velocity and angle of departure in the present case are, approximately,

$$\Delta V_G = G_X \cos \phi \quad (1114)$$

$$\sin \Delta \phi_G = \frac{G_X \sin \phi}{V} \quad (1115)$$

and that in the case of positive gun motion ΔV_G is positive and $\Delta \phi_G$ negative, while in the case of negative gun motion ΔV_G is negative and $\Delta \phi_G$ positive.

If we let ΔX_V denote the change in range due to ΔV_G , and ΔX_ϕ the change due to $\Delta \phi_G$, we may write

$$\Delta X_G = \Delta X_V - \Delta X_\phi \quad (1116)$$

in which ΔX_G takes the same sign as the component of gun motion itself. Comparing (1109) and (1116), we can establish a relation between ΔX_G and ΔX_W as follows. Substituting (1116) in (1109) we have

$$\Delta X_W = (\text{Col. 15}) - \Delta X_G$$

whence

$$\Delta X_G = (\text{Col. 15}) - \Delta X_W.$$

Since all of the range-table columns concerned are based on components of equal magnitude (10 knots), and since ΔX_W for a 10-knot component is given by Column 13, we may write finally

$$\Delta X_G \text{ (or Col. 14)} = (\text{Col. 15}) - (\text{Col. 13}). \quad (1117)$$

1118. Mathematical analysis therefore leads to the conclusion, which follows from a literal interpretation of formula (1117), that the effect of a given component of gun motion is the equivalent of the effect of an equal component of target motion less the effect of an equal component of wind. This conclusion is further substantiated by reasoning, as follows. The projectile, entirely apart from the velocity imparted to it by the charge in the gun, acquires from the moving gun a horizontal motion G_X which, if unopposed, would result in a change of range equal to $G_X \times T$ in a time of flight T ; and for equal components, $G_X \times T = T_X \times T = \text{Col. 15}$. But the motion thus acquired by the projectile is, in fact, opposed by air resistance, and the resultant change in range is accordingly less than $G_X \times T$ (or Col. 15). The effect of the air resistance upon the motion G_X initially acquired by the projectile may be regarded as the equivalent of the effect of an *apparent wind*, W_X , equal in magnitude but opposite in direction to G_X —that is, the equivalent of the value given in Column 13 of the range table.

The conception that the motion of projectile acquired from the moving gun is opposed by an apparent wind is more readily grasped in the case of gun motion across the line of fire, since in this case the projectile's motion as acquired from

Formula for
determination
of effect of
gun motion
on range

Analysis of
relation among
effects of gun
motion, wind,
and target motion

the motion of the gun can more readily be separated from its motion as acquired from the charge in the gun. Let us suppose, for example, that a gun is fired abeam to starboard from a ship steaming at 10 knots, and that there is no actual wind. Now although it has been specified that there is no *actual* wind, the projectile nevertheless will encounter an *apparent* wind of 10 knots from its left, just as a person standing on the ship and facing to starboard will feel an apparent wind of 10 knots from his left. Since the projectile, upon leaving the gun, has acquired a sidewise motion due to the motion of the gun across the line of fire, it will continue to experience an apparent wind equal in magnitude to and opposing this sidewise motion.

The relation among the changes in range due to gun motion (Col. 14), target motion (Col. 15), and wind (Col. 13), as expressed by formula (1117), should be borne in mind, for it has an important bearing on the practical use of these columns of the range table, as will be brought out in Chapter 12. The same applies to a similar relation among the lateral deviations due to gun motion (Col. 17), target motion (Col. 18), and wind (Col. 16), which will be dealt with shortly.

1119. The computation of values for Column 14 is illustrated in the following example.

Computation of Column 14	<p><i>Given:</i> In the computation of range-table values for the 16" 2600 f.s. gun, for $\phi = 25^\circ$, the values for Columns 13 and 15 have been found to be, respectively, 157 yards and 325 yards.</p> <p><i>Find:</i> The change in range due to a 10-knot component of gun motion in the line of fire (Col. 14).</p>
-------------------------------------	---

Applying formula (1117), we have

$$\Delta X_G = 325 - 157 = 168 \text{ yards.}^*$$

LATERAL DEVIATION DUE TO A GIVEN COMPONENT OF GUN MOTION PERPENDICULAR TO THE LINE OF FIRE (COLUMN 17).

1120. We shall denote by G_z the component of gun motion perpendicular to the line of fire (positive when to the right and negative when to the left with respect to the direction of fire), and by D_G the lateral deviation resulting from such motion. Figure 24 (b) can be used to represent the vector analysis for the case of gun motion, by substituting G_z for W_z . The resultant of the vectors OA and G_z is then OB , as before, and the essential difference between OA and OB is the lateral angular displacement of OB with respect to OA , which is measured in the horizontal plane by the angle $DOE = \gamma$. From the construction we have

$$\tan \gamma = \frac{G_z}{V \cos \phi}$$

**Formula for
determination
of lateral
deviation due
to gun motion**

and in a range X , this angular displacement will result in the lateral deviation

* The fact that Column 14, in the range tables, is not always exactly equal to the difference between Columns 15 and 13, is due to the fact that it has been customary to compute Column 14 from formulas (1114), (1115), and (1116). This should not of itself change the relation among the three columns, as stated above and as used here for computing Column 14; it does, however, occasion slight computational differences, and differences which arise from the smoothing operations which are applied separately to the three columns. On the whole, nothing at all is to be gained by the greater labor of computing Column 14 from the formulas. The same applies to the slight discrepancies that are found between Column 17 and the difference between Columns 18 and 16.

$$D_G = \frac{XG_z}{V \cos \phi} \quad (1118)$$

Comparing (1118) with (1112), and following through the steps already indicated in article 1117, we find

$$D_w = (\text{Col. 18}) - D_G$$

whence

$$D_G = (\text{Col. 18}) - D_w$$

and finally

$$D_G \text{ (or Col. 17)} = (\text{Col. 18}) - (\text{Col. 16}). \quad (1119)$$

Mathematical analysis therefore leads to the same conclusion in the case of gun motion *across* the line of fire as in the case of gun motion *in* the line of fire; logical support for this conclusion as applied to the present case has already been given in article 1118.

1121. The computation of values for Column 17 is illustrated in the following example.

Given: In the computation of range-table values for the 16"2600 f.s. gun, for $\phi = 25^\circ$, the values for Columns 16 and 18 have been found to be, respectively, 101 yards and 325 yards.

Find: The lateral deviation due to a 10-knot component of gun motion perpendicular to the line of fire (Col. 17).

Computation
of Column 17

Applying formula (1119), we have

$$D_G = 325 - 101 = \underline{224 \text{ yards.}^*}$$

EXERCISES

- Given:** The range, and the angle of departure and values from Columns 2 (b), 10, and 15 of the range table corresponding thereto.
Find: The change in range due to a 10-knot component of wind in the line of fire (Col. 13).

		Given					Answers
		X (yds.)	ϕ	Col. 2 (b)	Col. 10 (yds.)	Col. 15 (yds.)	ΔX_w (Col. 13) (yards)
16" 2600f.s.	A	24000	16° 07'	6.4	140	223	93
	B	30000	23 19	8.2	173	307	146
	C	32000	26 11	9.1	185	338	166
	D	38000	38 04	17.4	230	463	236
	E	39000	41 20	25.9	240	493	245

- Given:** The range, and the angle of departure and time of flight corresponding thereto.
Find: The lateral deviation due to a 10-knot component of wind perpendicular to the line of fire (Col. 16).

* See note adjacent to art. 1119.

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Given					Answers
		X (yards)	ϕ	T (sec.)	D_T (Col. 16) (yards)
A	5''3150 f.s.	5000	2° 00' 9	6.23	8
B		8000	4 14.0	12.02	25
C		10000	6 26.2	17.01	42
D	16''2600 f.s.	3000	1 17.9	3.61	1
E		5000	2 13.9	6.19	2
F		8000	3 47.9	10.31	6
G		12000	6 11.2	16.42	14
H		20000	12 14.8	30.96	41

3. *Given:* The range, and the time of flight corresponding thereto.

Find: The change in range and lateral deviation due, respectively, to 10-knot components of target motion in the line of fire (Col. 15), and perpendicular to the line of fire (Col. 18).

Given				Answers	
		X (yards)	T (sec.)	ΔX_T (Col. 15) (yards)	D_T (Col. 18) (yards)
A	5''3150 f.s.	5000	6.23	35	35
B		8000	12.02	68	68
C		10000	17.01	96	96
D	16''2600 f.s.	3000	3.61	20	20
E		5000	6.19	35	35
F		8000	10.31	58	58
G		12000	16.42	92	92
H		20000	30.96	174	174

CHAPTER 12

THE USE OF RANGE TABLES FOR THE CONTROL OF GUNFIRE FROM SHIPS. THE DETERMINATION OF BALLISTIC CORRECTIONS.

1201. From our previous study of naval gun sights (Chapter XII, *Naval Ordnance, 1933*), we have learned that the calibration of the range scale of a given gun is based on Columns 1 and 2 of that gun's range table. It is evident, then, that the gun, when properly aimed, will give the range set on its range scale only under the conditions which have been assumed for the computation of the range table, as defined in article 801. Under any other conditions the range given by the gun will vary from that set on its range scale, and in Chapters 10 and 11 we have learned how to determine the changes in range due to variations from any of the standard conditions set forth in article 801, except the last two (i.e., items (g) and (h)). With regard to item (h), viz., the assumption that the gun and target are in the same horizontal plane, it has already been shown in article 813 that the requirements of this condition are practically fulfilled when the gun is laid in elevation with respect to the line of sight, as is usually the case in naval gunnery; also, that if the gun should be laid with respect to the horizontal, as may be the case in indirect fire, the required corrections are obtainable from formulas (803) and (804), although such corrections are of small importance in comparison with other problems that enter into this type of fire. The single remaining assumption (item (g)), viz., that the earth is motionless, will be dealt with further in Chapter 13.

1202. The algebraic sum of the several changes in range due to variations from standard conditions, evidently determines the error in range that would result if the gun were fired with the actual target distance set on its range scale. In order to compensate for this error, we must therefore set the sights for a range which is the algebraic sum of the target distance and of these several changes in range. The range actually set on the gun's range scale, or the *sight-bar range*, then is an artificial value by means of which the gun's angle of elevation is adjusted in order that the gun may, under the non-standard conditions, give a range equal to the actual target distance. It is essential to a proper understanding of ballistic corrections to appreciate that the sight-bar range does not in any sense define the actual range of the trajectory. The following example will make this clear. Let us suppose that, for the firing of a 16"2600 f.s. battery, the actual target distance is known to be 10,000 yards, which, under standard conditions, requires that the guns be laid at an angle of elevation of $4^{\circ}56'.8$ (see 16"2600 f.s. table in *Range and Ballistic Tables, 1935*). Let us suppose further that due to variations from standard conditions the guns, if laid as indicated above, would give a range 500 yards short of the desired 10,000 yards ((-) 500 yards being the algebraic sum of the several changes in range, as determined from the range-table columns). From Column 2 (b) of the range table we find that in order to compensate for this reduction in range, we must increase the angle of elevation by $18'.0$; that is, we must set the sights for an angle of elevation of $4^{\circ}56'.8 + 18'.0 = 5^{\circ}14'.8$. The latter can be accomplished very simply, of course, by setting on the sights the sight-bar range 10,500 yards, i.e., the range-

table range which corresponds to the angle of elevation of $5^{\circ}14'8''$. The point to be grasped is that the guns will now give a range of 10,000 yards, and that the setting 10,500 on their sights is, in fact, only an artificial value by means of which the required angle of elevation of $5^{\circ}14'8''$ is obtained. This fact must be considered in connection with the choice of the range on which ballistic corrections are to be based, as will appear presently.

1203. The unqualified term *range* is commonly used to denote either distances, or settings on the sights of the gun, and ordinarily the proper sense of the term can be inferred from the connections surrounding its use. In our present discussions, however, it will frequently be necessary to distinguish specifically between range in the sense of distance, and range in the sense of a sight-setting, and we shall, for this purpose, use the following terms.

(a) *Target distance*. This denotes the actual distance from the gun to the target.

(b) *Sight-bar range*. This denotes the setting on the range scale of the gun, and it should equal the target distance only when the conditions of firing agree in all respects with (or are equivalent to) the standard conditions assumed for the range table.

In practice, various approximations of the target distance may be available, such as the *present range*, which is the best estimate of the target distance as determined at the time of firing, and the *navigational range*, which is the best estimate of the same distance as determined for post-firing analysis. Similar distinctions are made also in connection with the sight-bar range. Such distinctions are required chiefly for the purpose of post-firing analysis. It is to be understood that the terms defined above are general in nature; that is to say, the target distance may be merely an assumed value, or it may be based on a range-finder range taken just prior to firing, or it may be based on a careful plot of data available after the firing; and the sight-bar range, of course, partakes of the limitations of the target distance on which it is based.

1204. The algebraic sum of the drift and other lateral deviations (i.e., those due to wind, motion of gun, and motion of target), evidently determines the lateral error that would result if the bore of the gun were set in the vertical plane containing the gun and target. In order to compensate for this lateral error we must offset the bore with respect to this vertical plane, and we accomplish this by offsetting the line of sight laterally with respect to the bore so that when the line of sight is brought on the target the bore has the required lateral offset. For example, if the algebraic sum of all the lateral deviations is 100 yards to the right, the bore must be directed at a point 100 yards to the left of the target in order that the trajectory may terminate at the target; that is, the *line of sight* must be offset with respect to the *bore* by an angle which will cause the latter to point 100 yards to the left of the target when the line of sight is on the target. If the distance to the target is 10,000 yards, this angle then must be, in this case, the angle whose

tangent equals $\frac{100}{10,000}$; or, in any case, the line of sight must be offset laterally

with respect to the bore by an angle whose tangent equals the algebraic sum of all the lateral deviations divided by the target distance.

The lateral angle between the line of sight and the bore is called the *deflection*, and it is set by means of a deflection scale. (The mechanical arrangements

Deflection for setting deflection are shown in Chapter XII, *Naval Ordnance, 1933*.) For convenience, the deflection scale is graduated in terms of an arbitrary angular unit known as the *mil*, the latter being the angle which is subtended by an arc equal to one one-thousandth of the arc's radius, and hence being equal to .001 radian (approximately $3'44$, or $3'26''$). Within the limits of deflection that are likely to occur in practice, a very close approximation of the angular deflection corresponding to a given lateral deviation can then be obtained by dividing the latter by one one-thousandth of the target distance and using the result so obtained as the deflection in mils.* For example, let us suppose that the lateral deviation is 1000 yards, and the target distance 10,000 yards (which represents a rather extreme case). The deflection ac-

tually required in this case is $\tan^{-1} \frac{1000}{10,000} = 5^\circ 42'6$. If we use, instead, $\frac{1000}{10} = 100$

mils, we are then actually using a deflection of $100 \times 3'44 = 5^\circ 44'$, corresponding to a lateral deviation of 1004 yards at the given target distance of 10,000 yards. The error, even in this extreme case, is slight, and in ordinary cases it is practically insignificant. It may be stated, therefore, that *the deflection in mils is obtained by dividing the lateral deviation by one one-thousandth of the target distance*.

Deflection scales are marked so that parallelism between the line of sight and bore is indicated by the setting 100 (formerly the setting 50 was used to indicate parallelism, and in antiaircraft equipment the setting 500 is now so used), and so that the bore of the gun is directed to the *right* if the deflection setting is *raised* (i.e., numerically increased) and to the *left* if the deflection setting is *lowered* (i.e., numerically decreased), assuming that the line of sight remains pointed at the same object. An easily remembered rule for determining the proper direction in which to set the sights in deflection follows from the above; for in order to move the point of fall to the right we must raise the deflection setting (*right, raise*), and in order to move the point of fall to the left we must lower the deflection setting (*left, lower*). This rule applies, of course, to the direction in which the *correction* is to be made, and not to the direction in which the *error* is found. For example, if the algebraic sum of all the lateral deviations is 100 yards to the *right*, and the target distance is 10,000 yards, we evidently must apply a

correction of $\frac{100}{10} = 10$ mils to the *left*, and, in accordance with the above rule,

Definition of the term "scale" the required deflection setting is then $100 - 10 = 90$. For the sake of convenience and brevity in fire-control communication, the setting that is to be applied to the deflection scale is (in the U. S. Navy) always referred to as the *scale*; in the above example, then, the *scale* is 90.

* The mil is sometimes defined as being the angle whose tangent equals .001, which is practically equal to .001 radian. The U. S. Army uses a *mil* which is defined by the relation $6400 \text{ mils} = 360^\circ$, as compared with the corresponding relation $2000 \pi \text{ mils} = 360^\circ$, or $6283.2 \text{ mils} = 360^\circ$, which applies to the graduations of U. S. Navy deflection scales.

Formerly deflections scales in the U. S. Navy were graduated in *knots* instead of in mils, but this convention was discontinued at about the time of the World War. Very few scales of this type are still in service; an explanation of their use, if required, may be found in article 323, *The Groundwork of Practical Naval Gunnery, 1915*.

1205. In order that the sights of a gun may be set correctly, we must determine the effects of the variations from the several standard conditions upon which a range table is based (art. 801). The range-table columns give the *errors* in the point of fall that correspond to variations of given amounts in the several conditions for which standard values have been assumed. The errors may be in range, or they may be lateral.

We shall refer to errors in range as *range errors*, and to the corresponding corrections as *range corrections*. If the error is such as to increase the range, we shall call the *error over* and its corresponding *correction down*. If the error is such as to decrease the range, we shall call the *error short* and its corresponding *correction up*.

We shall refer to lateral errors (or deviations) in yards as *deflection errors in yards*, and the corresponding corrections as *deflection corrections in yards*. If the error is such as to move the point of fall to the right, we shall call the *error right* and its corresponding *correction left*. If the error is such as to move the point of fall to the left, we shall call the *error left* and its corresponding *correction right*.

1206. The algebraic sum of all the range errors, calling overs (+) and shorts (-), is the total range error, and the latter with sign reversed is called the *ballistic correction in range*. The algebraic sum of the target distance and the ballistic correction in range, is the *sight-bar range*.

The algebraic sum of all the deflection errors in yards, calling right (+) and left (-), is the total deflection error in yards, and the latter with sign reversed, and after having been converted to mils (by dividing it by one one-thousandth of the target distance in yards), is called the *ballistic correction in deflection*. The algebraic sum of 100 (or other setting which indicates zero deflection) and the ballistic correction in deflection, is the *scale*.

The ballistic correction in range is usually prefixed by Up or Down, as "Up 500," "Down 500," etc. Algebraic signs may also be used; for example, "Up 500" may also be stated as "+500," and "Down 500" as "-500." Similarly, the ballistic correction in deflection is usually prefixed by Right or Left, as "Right 5," "Left 5," etc. Or, with algebraic signs, "Right 5" may also be stated as "+5," and "Left 5" as "-5." Both of these ballistic corrections together are sometimes referred to simply as the *ballistic*. For example, we say the ballistic is "Up 500-Right 2," or "Down 500-Left 6," etc.

1207. As we proceed to the problem of determining the individual items that constitute the ballistic corrections in range and in deflection, a question arises as to the range on which the various corrections are to be based. Let us assume, for example, that a battery of 16"2600 f.s. guns is to be fired at a target which is at a distance of 31,200 yards, and under conditions which are in accord with the standards on which the range table is based with the exception that the initial velocity is 50 f.s. less than standard. We find from the 16"2600 f.s. range table that the velocity reduction of 50 f.s. causes a reduction in range of $5 \times 180 = 900$ yards, whence, in accordance with the rules already stated, we determine that a sight-bar range of $31,200 + 900 = 32,100$ yards is required. The question arises, in this case, whether the drift shall be found with the target distance* of 31,200 yards, or with the sight-bar-range 32,100 yards.

* It is to be understood, of course, that we are dealing here, as well as elsewhere throughout the following discussions, with the trajectory whose point of fall actually occurs at the

That the drift in this case cannot be found directly from the range table either with the target distance or with the sight-bar range, becomes evident when we consider that the required trajectory agrees neither with that corresponding to the range-table range of 31,200 yards, nor with that corresponding to the range-table range of 32,100 yards. By using the sight-bar range 32,100 yards we are giving the gun the angle of elevation of $26^{\circ}20'1''$, which corresponds to this range in the range table, but nevertheless we still expect the gun to give a horizontal range of only 31,200 yards. The trajectory which is required to give the horizontal range of 31,200 yards in this case is, in fact, one which corresponds to the arguments $\phi = 26^{\circ}20'1''$ and $V = 2550$ f.s., as compared with the range-table trajectory (for the same horizontal range) which corresponds to the arguments $\phi = 25^{\circ}00'0''$ and $V = 2600$ f.s. The drift that actually corresponds to the sight-bar range 32,100 yards can be found by solving the drift formula (901), using the values $X = 31,200$ yards, $\phi = 26^{\circ}20'1''$, and $\omega = 37^{\circ}48'$, the required value of ω having been found from the A.L.V.F. Tables* with the arguments $\phi = 26^{\circ}20'1''$, $V = 2550$ f.s., and $\text{Log } C = 1.12707$ (the $\text{Log } C$ for this example is the same as that already found in article 815). The solution shows that the drift in this case is 783.7 yards, as compared with the range-table values 739.7 yards corresponding to the target distance 31,200 yards, and 809.4 yards corresponding to the sight-bar range 32,100 yards.

By assuming other likely variations from standard conditions, we find a similar situation, namely, that the correct value of the drift cannot be obtained from the range table directly either with the target distance or with the sight-bar range, but that the correct value lies somewhere between those obtained from the range table, respectively, with the target distance and with the sight-bar range. Ordinarily a very good approximation of the drift can be obtained by entering the range table with the mean of the target distance and sight-bar range. In the above example this mean range is 31,650 yards, and the corresponding drift from the range table is 773.7 yards, which differs by less than 10 yards from the correct value.

It can likewise be shown that very good approximations of the lateral deviations due to wind, motion of gun, and motion of target are obtainable with the mean of the target distance and sight-bar range. For example, the time of flight corresponding to the conditions of the above example is found from the A.L.V.F. Tables (with $\phi = 26^{\circ}20'1''$ and $V = 2550$ f.s.) to be 59.62 seconds, which is greater than the range-table value of 57.79 seconds corresponding to the target distance 31,200 yards, but less than the range-table value of 60.39 seconds corresponding to the sight-bar range 32,100 yards. Column 18 depends directly on this T which lies between the values for target distance and sight-bar range. Column 16 depends on this T , and also on a value of X which is equal to the target distance and on a value of ϕ which corresponds to the sight-bar range. Column 17 is, in effect, a combination of Columns 16 and 18 (art. 1120). In all of these cases, therefore, the value of the effect must lie between the range-table values corresponding, respectively, to target distance and sight-bar range. The mean of target distance and sight-bar range, although less satisfactory for some of these columns than for others, as will be evident from the different relations upon which the several

target, that is to say, the trajectory whose horizontal range is equal to the target distance. Changes in the actual horizontal range of the trajectory should, for the purposes of these discussions, be regarded as the equivalent of changes in the target distance.

* It is necessary to use second differences in interpolating with respect to ϕ in these tables (see art. 712).

columns depend, nevertheless generally gives better approximations than are obtainable with the target distance or sight-bar range themselves.

1208. Similar reasoning is applicable also in the case of range errors. Let us continue with the example already stated above, viz., 16"2600 f.s. battery, target distance 31,200 yards, velocity reduction 50 f.s., sight-bar range 32,100 yards, and angle of elevation $26^{\circ}20'1$ corresponding to the latter. The change of range due to the 50 f.s. velocity reduction was determined from the value in Column 10 for the target distance 31,200 yards (i.e., $5 \times 180 = 900$ yards). The question now is whether we shall make a further approximation by taking the value from Column 10 for the sight-bar range 32,100 yards, which would then give a change in range of $5 \times 186 = 930$ yards, and hence a new sight-bar range of 32,130 yards. But the value $\Delta X_V = 930$ yards, as found by this process, corresponds to the arguments $\phi = 26^{\circ}20'1$ and $V = 2600$ f.s., whereas the arguments are actually $\phi = 26^{\circ}20'1$ and $V = 2550$ f.s. The new value of ΔX_V found with the sight-bar range is therefore too great, and the second approximation of the sight-bar range (i.e., 32,130 yards) is also too great. It will generally be the case for range, as well as for deflection, that the true value of a ballistic correction lies between the values found from the range table, respectively, with target distance and sight-bar range.

Considering that in any actual case the range is affected by a combination of variations in V and C and of wind, motion of gun, and motion of target, it would be an endless task to determine the relative merits of the use, respectively, of target distance and sight-bar range for finding individual ballistic corrections from the range table under the great variety of such combinations that is likely to occur in practice. A fairly comprehensive analysis based on likely combinations of variations in V and C was made recently,* and it showed that in some cases the use of the sight-bar range for finding ballistic corrections improves matters, while in other cases it makes matters worse (the analysis was applied to both range and deflection). It was shown, moreover, that it is exceedingly difficult to make generalizations as to the relative merits of the target distance and the sight-bar range as arguments for the determination of ballistic corrections from the range table, either for range or for deflection.

A further consideration that enters into the problem is the fact that the values in the range-table columns from which ballistic corrections are to be determined are themselves, in many cases, rather broad approximations. Column 12, for example, gives the *mean* of the changes in range due to plus and minus 10% variations in air density; the use of this column involves error both because the column is based on a mean as to the sign of the variation, and because variations to which the column is to be applied in practice may be much smaller than that on which the column is based. The use of Column 10 also involves errors of this nature (see arts. 1004 and 1009). This is true, to varying degrees, of all of the columns from which ballistic corrections are determined (except Column 6). In view of these limitations of the range-table columns themselves, as well as of the accuracy with which most of the physical values that are to be applied to them can be obtained in practice (e.g., air density, velocity variations, wind, target speed), and of the uncertainty that usually exists as to the correct determination of the target distance itself, it seems altogether illogical to resort to such elabora-

*Ref. Letter of Special Board on Naval Ordnance (Bureau of Ordnance file S 72 4/49/184, of 1 Feb. 1930).

tions in the determination of ballistic corrections as are involved in basing them on successive approximations of the sight-bar range. For post-firing analyses, which can be based on relatively more accurate data than are generally available at the time of firing, somewhat greater elaboration in the determination of ballistic corrections may be justifiable. But even under the conditions of a deliberate post-firing analysis, it is doubtful whether any useful end is served by attempting any refinement beyond that of determining the several *deflection* corrections with the mean of the correct target distance and the correct sight-bar range. Any theoretical improvement in the *range* corrections that can be gained by basing them on a mean between target distance and sight-bar range is generally of small consequence in comparison with the uncertainty that remains in the value of the target distance itself, even considering the best value of the latter that is obtainable for post-firing analysis.

1209. The best that can be said, in the way of a conclusion based on the above discussions, is that due caution must be exercised in laying down any general rules that are to govern the choice of the range argument on which ballistic corrections are to be based, and that attempts to improve the ballistic corrections by basing them on successive approximations of the sight-bar range may, in fact, serve merely to introduce additional error. A broad view of the situation, giving due weight to practical considerations as well as to theoretical considerations, leads to the conclusion that the preponderance of advantages lies on the side of the simplest practice of all, namely, that of basing all ballistic corrections on the *target distance*, and this practice will be followed in the examples given in this textbook. Various other practices for the determination of ballistic corrections are prescribed, from time to time, in the official instructions governing such matters, and the latter must be followed in the computation and analysis of ballistic corrections in connection with the preparation of target-practice reports.

It is to be noted, however, that no matter what practice is followed in determining the deflection error in yards, the conversion of the latter to mils must always be based on the *target distance*,* for the value of the mil is a function of actual distance and not of sight-bar range.

We shall now consider the various other details involved in the use of the range-table columns for determining the individual items that constitute the ballistic corrections in range and in deflection.

1210. Column 10 gives the change in range corresponding to a velocity variation of the magnitude stated in the heading of the column. Simple proportion may be used for finding changes of range corresponding to any velocity variations that are likely to be occasioned by the combined effects of erosion and variations in powder temperature, but this column is not applicable at all to the changes in velocity occasioned by reduced charges (arts. 1004 and 1006). The matter of signs has already been discussed in article 1002. There should be no confusion as to signs, for an *increase* in initial velocity always causes an *increase* in range, i.e., a *plus error*, or an *over*; and a *decrease* in initial velocity always causes a *decrease* in range, i.e., a *minus error*, or a *short*.

Variations from the standard initial velocity are due principally to two causes, viz.,

* See also note on page 158.

**Variations
in powder
temperature**

- (a) *Variations in the temperature of the powder.* The standard initial velocity is based upon the standard powder temperature of 90°F., and it is assumed that a variation of $\pm 1^\circ$ from this standard temperature causes a variation of ± 2 f.s. in the initial velocity, for all guns. This relation is an approximate one, but serves conveniently as an average for all guns and all likely temperature variations. Variations in excess of about 20°F. from the standard should be avoided, however, by control of magazine temperatures.

Erosion

- (b) *Erosion.* The gradual enlargement of the bore (especially at the origin), due to the wear incident to firing, results in a reduction of the initial velocity that is given by the standard charge. The amount of the velocity loss due to erosion is determined experimentally at the proving ground; it varies for different guns, and for a given gun it varies with the weight of charge. The proving ground furnishes *erosion curves* for the several guns and for their several designed charges (i.e., full and reduced charges). Two types of such curves are now generally furnished for the determination of velocity loss due to erosion, viz., (1) a plot of velocity loss against *bore enlargement*, and (2) a plot of velocity loss against *rounds fired*. Since bore enlargement is the immediate source of the velocity loss that is occasioned by erosion, the velocity-loss-bore-enlargement relation provides the more positive means for the determination of erosion losses. The velocity-loss-rounds-fired relation is necessarily less positive in its application to individual guns, since it does not provide for the differences in wear that the same number of rounds may cause in different guns of the same type. Guns should be star-gauged as frequently as practicable for the determination of bore enlargement. (This requires the services of equipment and specially trained personnel now carried only at navy yards and on some fleet tenders.) In the absence of up to date bore-enlargement data, the erosion loss should be carried forward from that indicated by the last star-gauging, by applying to the latter the additional loss indicated by the rounds-fired erosion curve for the number of rounds fired since that star-gauging.

The following example illustrates the use of Column 10.

Given: The 16"2600 f.s. gun; temperature of powder 82°F.; velocity loss due to erosion 28 f.s.

Find: From the range table, at 10,000 and 30,000 yards, the range errors due to the variations from the standard initial velocity.

Velocity variation due to powder = $(-)8 \times 2 = (-)16$ f.s.

Velocity variation due to erosion = $(-)28$ f.s.

Total velocity variation = $(-)44$ f.s.

From Column 10, at 10,000 yards,

$$(-)44/10 \times 67 = (-)295 \text{ yards}$$

From Column 10, at 30,000 yards,

$$(-)44/10 \times 173 = (-)761 \text{ yards}$$

1211. Column 11 gives the change in range corresponding to a variation in weight of projectile of the magnitude stated in the heading of the column. Simple proportion may be used for finding changes of range corresponding to any likely variations from the designed weight of the projectile on which the range table is based. The matter of signs has already been discussed in article 1016. It is to be noted that the sign of the range error is usually, but not always, contrary to the sign of the variation in projectile weight; this has already been explained in article 1019.

Occasion for the use of Column 11 arises but rarely, for variations in weight of projectile are held within narrow limits. This column has little practical use except for the purpose of determining the amount of dispersion in range that may be attributed to such variations in projectile weight as are allowed by manufacturing tolerances. For example,

Given: Assuming a tolerance of ± 6 lbs. in the manufacture of projectiles for the 16"2600 f.s. gun.

Find: From the range table, at 10,000 and 30,000 yards, the range errors corresponding to the maximum variation in projectile weight that is occasioned by the given tolerance.

From Column 11, at 10,000 yards,
 $(\pm)6/10 \times (\mp) 19 = \underline{(\mp) 11 \text{ yards}}$

From Column 11, at 30,000 yards,
 $(\pm)6/10 \times (\mp) 15 = \underline{(\mp) 9 \text{ yards}}$

1212. Column 12 gives the change in range corresponding to a variation of $\pm 10\%$ from the standard air density.* Simple proportion may be used for finding changes of range corresponding to any variations in air density that are likely to occur in practice (art. 1009). The matter of signs has already been discussed in article 1007. There should be no confusion as to signs, for an *increase* in air density always causes a *decrease* in range, i.e., a *minus error*, or a *short*; and a *decrease* in air density always causes an *increase* in range, i.e., a *plus error*, or an *over*. For reasons that have been outlined in articles 420-425, the ballistic density should be used in determining the effects of variations from the standard air density.

As has already been explained in article 1014, a table of multipliers (Table V, *Range and Ballistic Tables, 1935*) is provided for use with Column 12, these multipliers being simply the ratios between any given variation in air density and the tabular variation of 10% on which Column 12 is based. It is important to note that the arguments assumed for Table V are the *surface density factor* (δ) and the maximum ordinate, and that the value of a multiplier as given in the body of the table represents the resultant of the operations, (1) of converting the surface density factor (δ), by means of Table IV, to the ballistic density (δ_b) assumed to correspond to the given surface density and maximum ordinate; and (2) of finding the ratio between the variation expressed by this ballistic density and the 10% variation on which Column 12 is based (see example in article 1014). That is to say, the multipliers in Table V embody the

* Column 12 is also useful for determining the effects of variations in other factors which enter into the ballistic coefficient. Thus, for example, this column gives also the change in range due to a variation of $\pm 10\%$ in the coefficient of form. The sign, as given by the column for a density variation, is also correct for a variation in coefficient of form, since δ and i both appear in the denominator of the ballistic coefficient.

determination of the ballistic density by means of Table IV, and thus render unnecessary the use of the latter table. But if the ballistic density has already been determined from actual aloft soundings, it is then necessary to eliminate that portion of Table V which represents the operation of converting surface density to ballistic density, and this requires that the multiplier be found with the ballistic density and zero maximum ordinate (see art. 1015). The following rules may therefore be stated.

**Rules for
determination
of multiplier
for Column 12**

- I. *If only surface observations are available, the multiplier for Column 12 is determined by entering Table V with the surface density factor (δ) as obtained from Table III, and with the maximum ordinate corresponding to the given range.*
- II. *If the ballistic density has been determined from actual aloft observations (as by an aerological party), the multiplier for Column 12 is determined by entering Table V with this ballistic density and zero maximum ordinate.*

1213. The following examples illustrate the use of Column 12 under each of the procedures stated above.

I. *Given:* The 16"2600 f s. gun, surface air temperature 48°F., surface barometric pressure 30."55.

Find: From the range table, at 10,000 and 30,000 yards, the range errors for the ballistic densities which are *assumed* to correspond to the given surface conditions.

From Table III, with 48° and 30."55,

$$\delta = 1.059$$

From the range table (Col. 8),

At 10,000 yds., $y_s = 709$ ft.

At 30,000 yds., $y_s = 12,001$ ft.

From Table V,

For $\delta = 1.059$ and $y_s = 709$ ft., $M = (-).58^*$

For $\delta = 1.059$ and $y_s = 12,001$ ft., $M = (-).41^*$

From Column 12, at 10,000 yards,

$$(-).58 \times 165 = \underline{(-)96 \text{ yards}}$$

From Column 12, at 30,000 yards,

$$(-).41 \times 1160 = \underline{(-)476 \text{ yards}}$$

II. *Given:* The 16"2600 f.s. gun, and the following ballistic densities, as determined for the stated ranges by the aerological party; at 10,000 yards, $\delta_s = .944$; at 30,000 yards, $\delta_s = .972$.

Find: From the range table, at 10,000 and 30,000 yards, the range errors corresponding to the given ballistic densities.

From Table V, entering with $y_s = 0$ in both cases,

For $\delta_s = .944$, $M = (+).56$

For $\delta_s = .972$, $M = (+).28$

* It is to be noted that the equivalent of three decimal places in the density factor is given by two decimal places in the multiplier; the latter therefore should not be carried beyond two places.

CHAPTER TWELVE

From Column 12, at 10,000 yards,

$$(+).56 \times 165 = (+)92 \text{ yards}$$

From Column 12, at 30,000 yards,

$$(+).28 \times 1160 = (+)325 \text{ yards}$$

It is to be noted that the sign of the multiplier for Column 12, as given by Table V, always denotes the sign of the range error corresponding to the conditions to which the multiplier pertains (art. 1015). The plus sign is, of course, to be understood when no sign is tabulated.

1214. Columns 13, 14, and 15 give the changes in range corresponding, respectively, to 10-knot* components of wind, motion of gun, and motion of target in the line of fire. Columns 16, 17, and 18 give the lateral deviations corresponding, respectively, to 10-knot* components of wind, motion of gun, and motion of target perpendicular to the line of fire. Simple proportion may be used for finding the changes in range or lateral deviations corresponding to any values of these components that are likely to occur in practice.

With regard to the signs that are to be used with these columns, it is evident that in the case of wind and motion of gun the direction of the error agrees with the direction of the component causing it. That is to say, wind or motion of gun in the direction of fire causes *plus errors*, or *overs*; wind or motion of gun contrary to the direction of fire causes *minus errors*, or *shorts*; and wind or motion of gun to the *right* or *left* with respect to the direction of fire causes, respectively, *plus errors (rights)* or *minus errors (lefts)*. In the case of target motion, however, the direction of the error is opposite to the direction of the component causing it. Target motion in the direction of fire causes *minus errors (shorts)*; target motion contrary to the direction of fire causes *plus errors (overs)*; and target motion to the *right* or *left* with respect to the direction of fire causes, respectively, *minus errors (lefts)* or *plus errors (rights)*.

1215. Since Columns 13-18 are based on components in the line of fire and perpendicular to the line of fire, it is necessary, in any case, to resolve the wind, motion of gun, and motion of target into such components. This can be done very simply, as illustrated in Figure 27. In this figure the position of the firing vessel is at *A*, and its course and speed are represented by the vector $G = AB$; the position of the target is at *C*, and its course and speed are represented by the vector $T = CD$; the direction and velocity of the wind are represented by the vector $W = EF$. The line of fire is AC ; θ_G is the angle between the firing ship's course and the line of fire, θ_T the angle between the target's course and the line of fire, and θ_W the angle between the direction of the wind and the line of fire. G , T , and W have, respectively, the components G_x , T_x and W_x in the line of fire, and G_z , T_z , and W_z perpendicular to the line of fire. It will be clear from the figure that $G_x = G \cos \theta_G$, $T_x = T \cos \theta_T$, and $W_x = W \cos \theta_W$; in other words, the components in the line of fire are in each case equal to the vector times the *cosine* of the angle included between the vector and the line of fire. Also, $G_z = G \sin \theta_G$, $T_z = T \sin \theta_T$, and $W_z = W \sin \theta_W$, or the components across the line of fire are in each case equal to the vector times the *sine* of the angle included between the vector and the line of fire.

* In the older range tables these columns were based on 12-knot components, instead of 10-knot components.

A convenient procedure, then, is to measure always the angles included between the directions of firing ship, target, and wind, and of the line of fire, for the components in the line of fire (range components) may then always be found with the cosines of these angles, and the components perpendicular to the line of fire (deflection components) always with their sines. For the purpose of determining the proper signs of the several components, no better or more convenient rule can be offered than

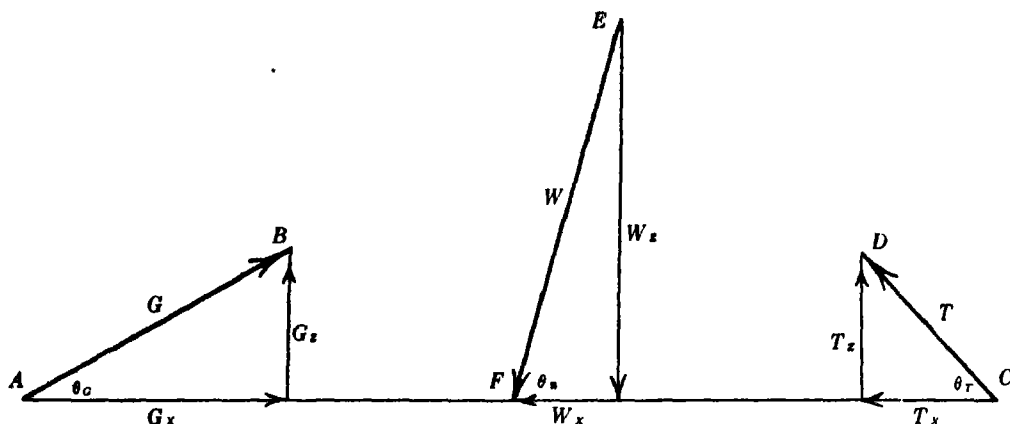


FIGURE 27

to draw a diagram similar to that in Figure 27; this avoids the troublesome procedure of deducing the signs of the components from the signs of the trigonometric functions on which they depend, and hence of considering the quadrants in which the several angles may lie.

1216. Wind, motion of gun, and motion of target are always measured in *knots* at sea. Hence we shall always express the vectors W , G , and T in knots. Let us assume for the vectors and included angles of Figure 27 the following values:

$$W = 20, \theta_W = 75^\circ; \quad G = 15, \theta_G = 30^\circ; \quad T = 10, \theta_T = 45^\circ.$$

Applying the rule stated above, and referring to Figure 27 for the directions of the components, we have

$$W_x = 20 \cos 75^\circ = 5.2 \text{ Against or } (-)$$

$$W_z = 20 \sin 75^\circ = 19.3 \text{ Right or } (+)$$

$$G_x = 15 \cos 30^\circ = 13.0 \text{ With or } (+)$$

$$G_z = 15 \sin 30^\circ = 7.5 \text{ Left or } (-)$$

$$T_x = 10 \cos 45^\circ = 7.1 \text{ Against or } (-)$$

$$T_z = 10 \sin 45^\circ = 7.1 \text{ Left or } (-)$$

The directions stated are the directions of the components themselves with respect to the direction of fire, and not the directions of the resultant errors in the fall of shot.

The several errors in the point of fall, for the 16"2600 f.s. gun at 10,000 yards, are now found as follows.

From	Col. 13, with $W_x = 5.2$; $\frac{5.2}{10} \times 19 =$ <u>10 yds. short or (-)</u>
	Col. 14, with $G_x = 13.0$; $\frac{13.0}{10} \times 58 =$ <u>75 yds. over or (+)</u>
	Col. 15, with $T_x = 7.1$; $\frac{7.1}{10} \times 75 =$ <u>53 yds. over or (+)</u>
From	Col. 16, with $W_z = 19.3$; $\frac{19.3}{10} \times 10 =$ <u>19 yds. right or (+)</u>
	Col. 17, with $G_z = 7.5$; $\frac{7.5}{10} \times 65 =$ <u>49 yds. left or (-)</u>
	Col. 18, with $T_z = 7.1$; $\frac{7.1}{10} \times 75 =$ <u>53 yds. right or (+)</u>

It will be observed that the several errors agree as to sign with their respective components except in the case of target motion, in which the signs of the errors are contrary to those of the components (art. 1214).

1217. In referring to wind we have heretofore assumed a *true wind*, i.e., a wind whose velocity and direction are measured with respect to the ground; the values in Columns 13 and 16 are based on such a wind. However, the wind that is actually felt and directly measured on a moving ship is an *apparent wind* which is the resultant of the true wind and of the ship's motion (and hence of the gun's motion). The course and speed of the ship being known, it is a simple problem in the resolution of forces to determine true wind from apparent wind, and the problem can be solved in various ways, of which a graphical solution is generally the most convenient. Mechanical devices based on the graphical solution are sometimes available, or they can be constructed without difficulty. Table 32, *Bowditch*, converts apparent wind to true wind, but not in units convenient for our present purposes.

It is sometimes convenient to base ballistic corrections directly on the apparent wind, i.e., without first converting the latter to true wind, and this can be done readily, as will appear from the following. It has already been shown (arts. 1117, 1118, 1120) that the effect of a given component of gun motion is the equivalent of the effect of an equal component of target motion* less the effect of an equal component of wind. It is evident, then, that the correction for a given component of gun motion differs from that for an equal component of target motion only in that it takes into account the effect on the trajectory of the wind which is set up by the gun's motion (the target's motion, of course, does not affect the trajectory itself). But the wind that is set up by the gun's motion (and hence ship's motion) is the same as that which, in combination with the true wind, makes up the apparent wind which is felt on the ship. It follows, then, that if we are to base the ballistic correction for wind on the *apparent wind*, we must omit from the ballistic correction for gun motion that portion which is designed to account for the wind set up by the gun's motion, since the latter has already been accounted for by being included in the apparent wind. But we have found above that a correction for gun motion differs from a correction for target motion only in that

* For the purposes of this relation, the effect of target motion is to be considered as being of the same sign as the effect of gun motion.

the former includes the effect of the wind set up by the gun's motion; that is to say, a ballistic correction for gun motion which omits this wind effect is the equivalent of a ballistic correction for target motion. Hence it follows that if the ballistic correction for wind is to be based on the apparent wind, the ballistic correction for gun motion must be based on values designed to apply to target motion.

1218. The conclusion stated above can be verified as follows. Let us assume that a ship is steaming on course north at a speed of 10 knots, that a true wind is blowing from north at a velocity of 10 knots, and that the battery of 16"2600 f.s. guns is to be fired abeam to starboard at a stationary target distant 10,000 yards from the ship. We find from Column 16 of the range table that the lateral deviation due to the 10-knot component of *true wind* blowing from the left is 10 yards to the right; and from Column 17 that the lateral deviation due to the 10-knot component of gun motion to the left is 65 yards to the left; hence the resultant lateral deviation is 55 yards to the left. Now the lateral deviation due to gun motion which we have found from Column 17 (i.e., 65 yards) is, in fact, based on the consideration that the 10-knot sidewise motion imparted to the projectile by the moving gun would amount to a lateral deviation of the projectile, during the time of flight, of 75 yards (Col. 18), provided that this sidewise motion were not opposed by the air; and on the further consideration that this sidewise motion is, however, opposed by the 10-knot wind which this sidewise motion creates against itself, and that the lateral deviation is therefore reduced by 10 yards (Col. 16), whence the resultant deviation becomes $75 - 10 = 65$ yards (which agrees with the value found from Column 17). But under the given condition of a *true wind* of 10 knots, the *apparent wind* is now 20 knots, and the projectile will experience this 20-knot apparent wind in its flight. For if we resolve the lateral motion of the projectile as before we find, in this case, that the 10-knot sidewise motion initially imparted to the projectile by the motion of the gun is now opposed not only by the 10-knot wind which this sidewise motion creates against itself, but also by the 10-knot true wind which is blowing, that is, altogether, by an apparent wind of 20 knots. Then the 75-yard deviation which the projectile would suffer if its sidewise motion were unopposed is, in this case, reduced by the effect of a 20-knot opposing wind, or by 20 yards (Col. 16), whence the resultant deviation is $75 - 20 = 55$ yards, which agrees with the result previously found. In the former case we have used the 10-knot component of true wind with Column 16 and the 10-knot component of gun motion with Column 17; in the latter case we have used the 20-knot component of apparent wind with Column 16 and the 10-knot component of gun motion with Column 18; the same result has been obtained in both cases. And what has been found here to be true for components perpendicular to the line of fire is likewise true for components in the line of fire.*

1219. The following rules may therefore be stated:

- I. *If true wind is to be used in determining the ballistic corrections, the range-table columns are to be used as follows:*

Rules for
use of true
wind and ap-
parent wind

For wind Columns 13 and 16
For motion of gun Columns 14 and 17
For motion of target Columns 15 and 18

- II. *If apparent wind is to be used in determining the ballistic corrections, the range-table columns are to be used as follows:*

<i>For wind</i>	<i>Columns 13 and 16</i>
<i>For motion of gun</i>	<i>Columns 15 and 18</i>
<i>For motion of target</i>	<i>Columns 15 and 18</i>

1220. The meaning and determination of the *ballistic wind* have already been explained in articles 1114–1116. The process of determining the components of wind in and perpendicular to the line of fire and applying them to the range-table columns, is the same whether the wind is a ballistic wind or merely a surface wind. The greater accuracy that is afforded by the use of the ballistic wind rather than merely the surface wind depends entirely on the fact that the former is a better measure of the wind itself than is the latter, and not on any difference as to the use of these two kinds of wind in connection with the range table.

The aerological party determines the *ballistic true wind*, i.e., a ballistic wind measured with respect to the ground. The *ballistic apparent wind* can be determined from the *ballistic true wind* by combining the latter vectorially with the true course and speed of the firing ship. For example, if the ballistic true wind is 10 knots from north, and if the firing ship is steaming at 10 knots on course north, the ballistic apparent wind is 20 knots from north. The use of true wind (whether ballistic wind or surface wind) is preferable to the use of apparent wind, since the apparent wind changes with every change in course or speed of the firing ship, whereas the true wind is not affected by the maneuvers of the firing ship. But many range-keepers are so designed that it is necessary to use with them ballistic corrections based on apparent wind,* and it should be borne in mind, when such range-keepers are used, that the ballistic true wind determined by the aerological party must be converted to the corresponding ballistic apparent wind by the firing ship itself.

1221. It is sometimes convenient, especially at target practices, to use as point of aim some well-defined feature of the target which is sufficiently displaced from the desired point of impact to require the application of a *point of aim correction*. When the latter is designed to compensate for a displacement in the point of aim in elevation, it is called the *point of aim correction in range*; and when designed to compensate for a displacement in the point of aim in train, it is called the *point of aim correction in deflection*. The following examples illustrate the determination of point of aim corrections.

I. *Given*: A night target practice is to be fired by a 5"3150 f.s. battery at ranges varying from about 4500 yards to about 2500 yards; the allowed target area is to be 20 feet high, and is to be represented by the lower half of a target which is 40 feet high. The point of aim is to be at the top edge of the whole target.

Find: From the range table, the point of aim corrections in range at 4500 yards and at 2500 yards.

The allowed target has a height of 20 feet, and its center, which is the desired point of impact, is therefore 10 feet above the water. The point of aim, being at the top of the whole target, is 40 feet above the water, or 30 feet above the desired point of impact. From Column 19 of the range table we find that the following changes in the sight-bar range are required to lower the point of impact 30 feet.

* This is the case with any range-keeper which corrects for motion of gun on the basis of Columns 15 and 18 of the range table, as will be evident from the rules stated in art. 1219.

$$\text{At 4500 yards, } \frac{(-) 30}{12} \times 100 = \underline{(-) 250 \text{ yards.}}$$

$$\text{At 2500 yards, } \frac{(-) 30}{5} \times 100 = \underline{(-) 600 \text{ yards.}}$$

The results found, viz., $(-)$ 250 yards and $(-)$ 600 yards, are the point of aim corrections in range, respectively, at 4500 yards and at 2500 yards;* it is to be noted that the signs as given pertain to the *corrections*, and not to errors.

II. *Given:* A target practice is to be fired by a 16"2600 f.s. battery at a range of about 25,000 yards; the target is to be 140 feet long and 40 feet high, and it is expected that it will lie practically perpendicular to the line of fire; the point of aim in train is to be at the left-hand edge of the target.

Find: The point of aim correction in deflection at 25,000 yards. Also determine from the range table how much difference in range is occasioned by shifting the point of aim in elevation from the top edge of the target to the bottom edge (or waterline).

The point of aim in train is $140/2 = 70$ feet, or about 23 yards, to the left of the desired point of impact. At 25,000 yards 1 mil = 25 yards, hence the point of aim *correction* in deflection is practically right 1 mil.

From Column 19 of the range table we find that the difference in range occasioned by shifting the point of aim in elevation 40 feet amounts to about

$$\frac{40}{140} \times 100 = \underline{29 \text{ yards.}}$$

1222. Column 19 can also be used for finding the height of impact above the waterline, in the vertical plane of the target, that corresponds to a given distance of point of fall beyond the target, or vice versa. For example, in the case of the 16"2600 f.s. gun at 10,000 yards, a shot which falls 120 yards beyond the target should have passed through the vertical plane of the target at a height of about

$$\frac{120}{100} \times 31 = 37 \text{ feet above the waterline. Or, if the target is 40 feet high, a shot}$$

which passes through the top of the target should fall at a distance of about

$$\frac{40}{31} \times 100 = 129 \text{ yards beyond the target.}^\dagger$$

1223. We have now examined the details involved in the use of the several range-table columns which are provided for the determination of the effects of

* Although it is not uncommon practice to use the top of the 40-foot target as the point of aim in elevation, it is often overlooked that at the close ranges which are likely to occur at night firings this may often require the use of sight-bar ranges which are materially less than the target distance.

† At very short ranges, such as occur at Short Range Battle Practice, the use of Column 19 for such purposes is subject to considerable inaccuracy. This will be dealt with in detail in Appendix B.

Practical methods for determining ballistic corrections

variations from the standard conditions on which the range table is based. Our next step will be to examine the principal features of practical methods by means of which all of these details may be handled expeditiously in the determination of the sight-bar range and scale. In actual practice, some of the elements of the ballistic corrections in range and in deflection are usually handled automatically by mechanical devices, such as the range-keeper; or some of them may be determined from graphical devices, such as alinement charts. However, intelligent use of such devices generally demands a thorough understanding of the entire problem of determining the several ballistic corrections on which the sight-bar range and scale depend. For this reason, if for no other, it is advantageous to acquire a close familiarity with the direct solution by computation of the several elements which enter into the determination of the sight-bar range and scale. Moreover, a post-firing computation of all elements of the ballistic corrections is required to be submitted with each target-practice report, whether or not some or all of these elements were determined at the time of firing by mechanical or graphical devices.

1224. A standard form, known as Sheet 10 of the target-practice report forms, is provided for the computation of ballistic corrections in connection with the preparation of target-practice reports. As a target-practice report form, Sheet 10 constitutes a post-firing analysis of the ballistic corrections in range and in deflection, and includes a complete record of the computations of the several elements composing these corrections and of the various conditions on which they depend. However, this sheet serves also as a convenient form for the pre-firing computation of those portions of the ballistic corrections which are not handled by the range-keeper or other devices, and it is often so used. In any event, it constitutes a summary of the various elements composing the ballistic corrections, and a guide as to the provisions which exist in various range-keepers for the automatic determination and application of some of these elements.

Since the notation and arrangement of data on Sheet 10 are sometimes confusing to the beginner, we shall use, for the illustrative problems that are to follow, a form in which the principal features of Sheet 10 are outlined in a somewhat more elementary fashion than on the sheet itself. The details involved in the various operations expressed on Sheet 10 will be obvious from an inspection of the form of solution given here.*

1225. A space is provided at the side of Sheet 10 for drawing a diagram representing the direction and speed of own ship and target, and the direction and velocity of the wind. This diagram is constructed by representing the motions of own ship, target, and wind as vectors with respect to the line of fire. The line of fire is represented by a line running from left to right across the diagram, and the own-ship vector is drawn at the left-hand end of the diagram, the target vector at the right-hand end, and the wind vector in the middle. We shall now examine the details involved in constructing the several vectors.

1226. The own-ship vector must make with the line of fire the angle between the direction of motion of own ship and the direction of the line of fire. This is equal to the *relative bearing of the target*. Usually we know this relative bearing directly, from observation. If we know, instead, the *true bearing*

Own-ship vector

* A sample of Sheet 10, and further details concerning its use, are not given here, because the sheet is changed from time to time. It is expected, however, that copies of the current Sheet 10, together with appropriate supplemental instructions relative to details which are not covered here, will be issued to midshipmen for study and for the solution of problems.

of the target, we must subtract therefrom the true course of own ship to obtain the relative bearing of the target; if this operation results in a negative angle, we must add 360° to the latter to get the corresponding positive angle. (Relative bearing is always measured clockwise, from ahead, through 360° .) To draw the own-ship vector on the diagram we must lay off the whole angle of relative bearing *anticlockwise* from the line of fire. It is a great aid in checking the correctness of the diagram to imagine oneself located at the own-ship origin and looking down the line of fire. The following examples illustrate the features just discussed:

- (a) Relative bearing of target = 45°
Own ship's speed = 12 knots

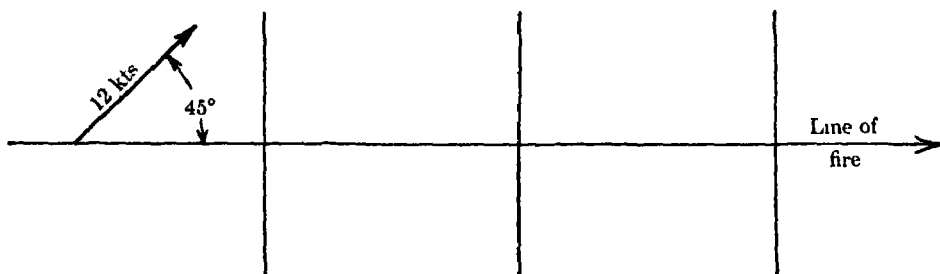


FIGURE 28

- (b) True bearing of target = 50°
Own ship's true course = 200°
difference = $(-)\overline{150^\circ}$
 $(+)\overline{360^\circ}$
Relative bearing of target = 210°
Own ship's speed = 12 knots

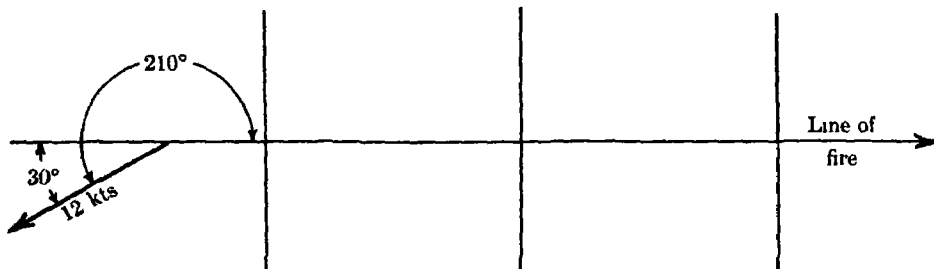


FIGURE 29

1227. The target vector must make with the line of fire the angle between the direction of motion of the target and the direction of the line of fire. This is equal to the *target angle*. We often know the target angle directly, as for example when the spotter observes it. If the target angle is not directly known but the target's true course is known, we must determine the tar-

Target
vector

get angle by referring the target's true course to the true direction of the line of fire. It is to be noted that the target angle is actually the relative bearing of own ship from the target. The operation of deducing target angle from true course and bearing is therefore similar to that of deducing relative bearing of target from true course and bearing. In the present case we must subtract the *target's true course* from the *reverse of the true bearing of the target*, adding 360° , if necessary, to get a positive angle. (Target angle, like relative bearing, is always measured clockwise, from ahead, through 360° .) To draw the target vector on the diagram we must lay off the whole target angle *anticlockwise* from the *line of fire reversed*. In plotting target angle it is a great aid to have in mind the conception of looking from the target back toward own ship. Examples:

- (a) Target angle $= 45^\circ$
 Target's speed $= 15$ knots

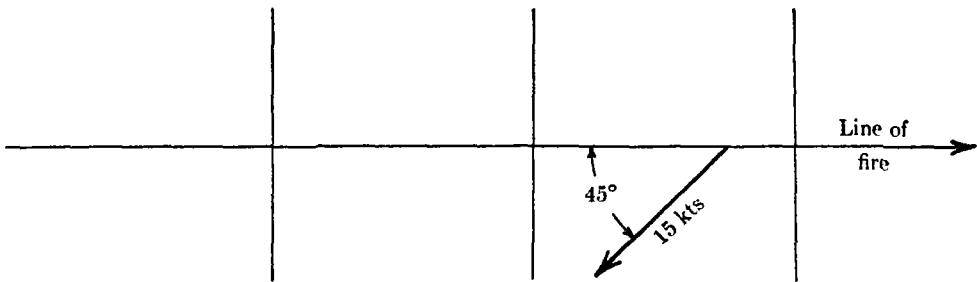


FIGURE 30

- | | | |
|-----------------------------------|---|------------------|
| (b) True bearing of target | = | 230° |
| Reverse of true bearing of target | = | 50° |
| True course of target | = | 200° |
| difference | = | $(-) 150^\circ$ |
| | | $(+) 360^\circ$ |
| Target angle | = | 210° |
| Target's speed | = | 15 knots |

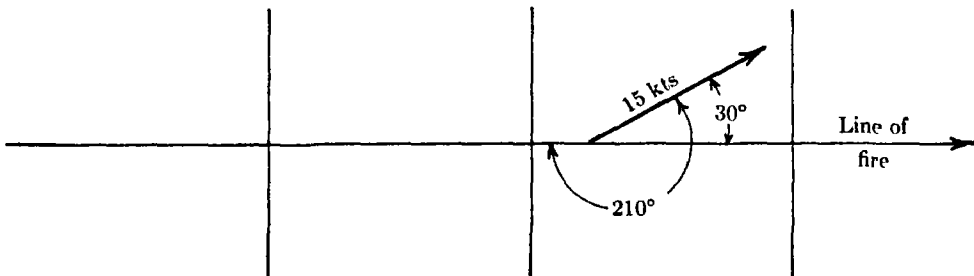


FIGURE 31

1228. The wind vector must make with the line of fire the angle between the direction *toward* which the wind is blowing, and the direction of the line of fire;

Wind vector this is known as the *wind angle*. The wind angle is measured clockwise, from the direction *toward* which the wind is blowing, through 360° . However, it is the universal practice in stating wind directions to state the direction *from* which the wind blows (e.g., a north wind blows *from* the north, not *toward* the north). This must be borne carefully in mind when plotting the wind vector. The term *bearing of the wind* is often used (as for example on sheet 10), and it also signifies the direction *from* which the wind blows. The wind angle is therefore determined by subtracting the *reverse of the bearing of the wind* from the *bearing of the target*, adding 360° , if necessary, to get a positive angle. To draw the wind vector we must lay off the whole wind angle *anti-clockwise* from the line of fire. The bearing of the wind may, of course, be either a true bearing or a relative bearing, and it is immaterial whether the wind angle is found from the relative bearings of wind and target or from their true bearings, but care must be taken not to use true bearing of wind with relative bearing of target, or vice versa. The relative bearing of wind corresponding to a given true bearing of wind can, of course, be obtained by subtracting from the latter the true course of own ship. Care must also be taken, when dealing with wind, not to confuse the term *true* as referred to bearing, with the term *true* as referred to the kind of wind being considered. The term *true* as opposed to the term *relative* relates to the *bearing* of the wind; the term *true* as opposed to the term *apparent* relates to the *kind* of wind being considered. The bearing of either kind of wind can be expressed either in true bearing or in relative bearing. Examples:

(a) Relative bearing of wind	=	90°
Relative bearing of target	=	45°
Reverse of relative bearing of wind	=	270°
difference	=	$(-) 225^\circ$
	=	$(+) 360^\circ$
Wind angle	=	135°
Wind velocity	=	10 knots

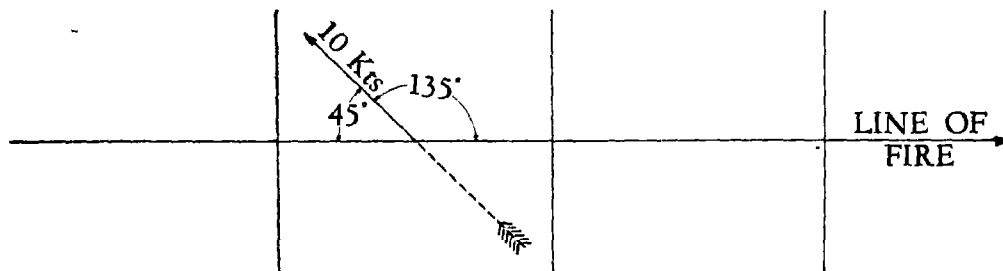


FIGURE 32

(b) True bearing of wind	=	10°
True bearing of target	=	340°
Reverse of true bearing of wind	=	190°
Wind angle	=	150°
Wind velocity	=	10 knots

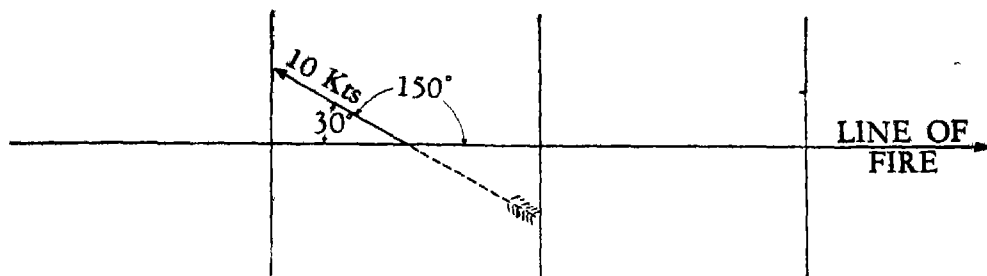


FIGURE 33

1229. The complete diagram is a graphic representation of the fire-control set-up that enters into the determination of the ballistic corrections. By drawing the vectors to scale, the components in and perpendicular to the line of fire can be measured graphically, and their computed values thus checked approximately. More important, however, is the aid that the diagram lends in determining the directions of the several components and of their corresponding errors. The following example illustrates these features.

Construction
and use of
the complete
diagram

Relative bearing of target	= 315°
Own ship's speed	= 18 knots
Target angle	= 120°
Target's speed	= 20 knots
Relative bearing of wind	= 165°
Velocity of wind	= 15 knots

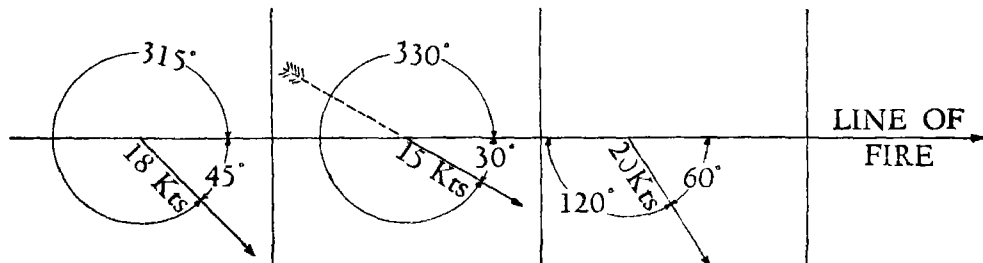


FIGURE 34

The diagram shows that own-ship motion is toward the target and to the right across the line of fire; the corresponding errors are *over* in range and *right* in deflection. Target motion is away from own ship and to the right across the line of fire; the corresponding errors are *short* in range and *left* in deflection. The wind is blowing toward the target and to the right across the line of fire; the corresponding errors are *over* in range and *right* in deflection. It is to be noted that although all three vectors in the above example have range components pointing down the line of fire and deflection components pointing to the right of the line of fire, the errors due to own ship and wind are *over* and *right*, while those due to the target are *short* and *left*; the reason for this has already been given in article 1214.

It will be observed, in the examples given above, that in all cases where the whole angle used in plotting the vector lies in the second, third, or fourth quadrant, the diagram shows not only the whole angle but also the corresponding first-quadrant angle, for the latter can be used more conveniently for finding the required trigonometric functions. In actual practice it is customary to label on the diagram only these first-quadrant angles.

1230. Some prefer, when dealing with true courses and bearings, to make a navigational plot of the set-up, since this kind of diagram serves not only for the purpose of showing the vectors in relation to the line of fire, but also for the purpose of reducing to a simple graphical process the entire problem of determining the required angles. Figure 35 illustrates the application of this method to the following problem.*

Navigational
diagram

True course of own ship	=	30°
Speed of own ship	=	15 knots
True bearing of target	=	60°
True course of target	=	285°
Speed of target	=	10 knots
True bearing of wind	=	345°
Velocity of wind	=	20 knots

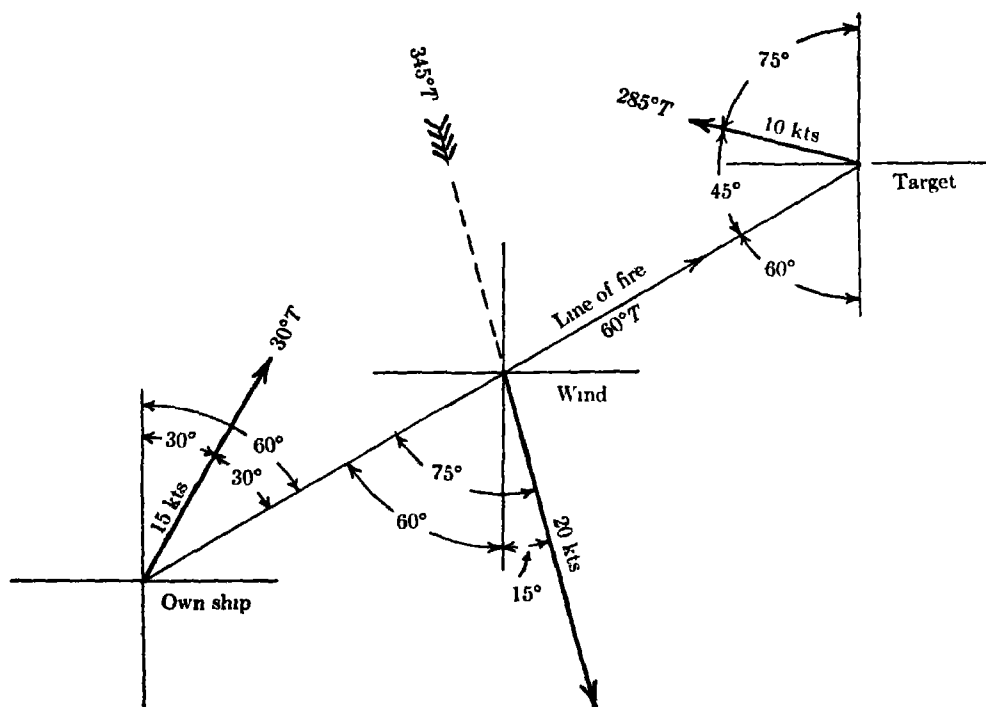
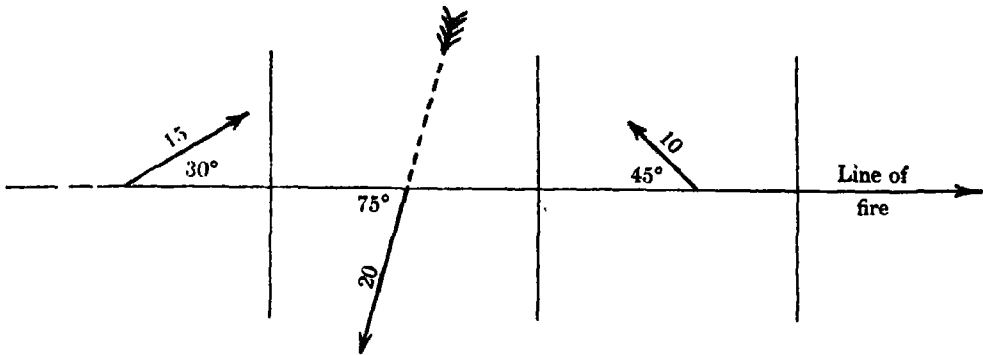


FIGURE 35

* The angles determined from a diagram of this type can, of course, be transferred without difficulty to the type of diagram which is provided for at the top of Sheet 10. The problem in article 1231, being based on the same set-up as stated for the present problem, affords a comparison between the two types of diagram.



16" 2600 f.s. Gun
Target Distance 30,000 yds.

				Range Errors		Deflection Errors	
				Over	Short	Left	Right
Col. 6 656						×	656
Col. 10 173	Erosion Powder temp 82°F., (-) 8 × 2 = Tot. vel. var.	- 28 f.s. - 16 f.s. (-) 44 f.s.	$\frac{44}{10} \times 173 = 761$	×	761		
Col. 11 15	Var. from standard weight of projectile	0	$\frac{0}{10} \times 15 = 0$	×	×		
Col. 12 1160	Air temp. 48°F. Barometer 30."55 s (T. III) 1.059 Max. ord. 12,001 ft. Multiplier (T.V) (-) .41		$.41 \times 1160 = 476$	×	476		
Col. 13 145	Wind velocity (true) ✕ with line of fire 20 cos 75° = 5.2	20 75°	$\frac{5.2}{10} \times 145 = 75$	×	75		
Col. 14 160	Own ship's speed ✕ with line of fire 15 cos 30° = 13.0	15 30°	$\frac{13.0}{10} \times 160 = 208$	208	×		
Col. 15 307	Target's speed ✕ with line of fire 10 cos 45° = 7.1	10 45°	$\frac{7.1}{10} \times 307 = 218$	218	×		
Col. 16 94	Wind velocity (true) ✕ with line of fire 20 sin 75° = 19.3	20 75°	$\frac{19.3}{10} \times 94 = 181$			×	181
Col. 17 212	Own ship's speed ✕ with line of fire 15 sin 30° = 7.5	15 30°	$\frac{7.5}{10} \times 212 = 159$			159	×
Col. 18 307	Target's speed ✕ with line of fire 10 sin 45° = 7.1	10 45°	$\frac{7.1}{10} \times 307 = 218$			×	218
Total range error..... 886 yds. Short				426	1312	159	1055
Total deflection error..... 896 yds. Right					426		159
1 mil = $\frac{30,000}{1000} = 30$ yds., $\frac{896}{30} = 30$					886		896

Ballistic Correction in Range..... Up 886 yds.
Ballistic Correction in Deflection... Left 30 mils
Sight-bar Range 30,900 Scale 70

1232. Given: The 5''3150 f.s. gun and its range table; target distance 8,500 yards; powder temperature 79°F.; velocity loss due to erosion 44 f.s.; ballistic density .933 (as determined by the aerological party); relative bearing of target 322°; own ship's speed 18 knots; target angle 22°; target's speed 33 knots; relative bearing of ballistic *apparent* wind 10°, velocity 32 knots; average weight of projectiles 48 lbs.

Find: The ballistic corrections in range and in deflection, and the sight-bar range and scale.

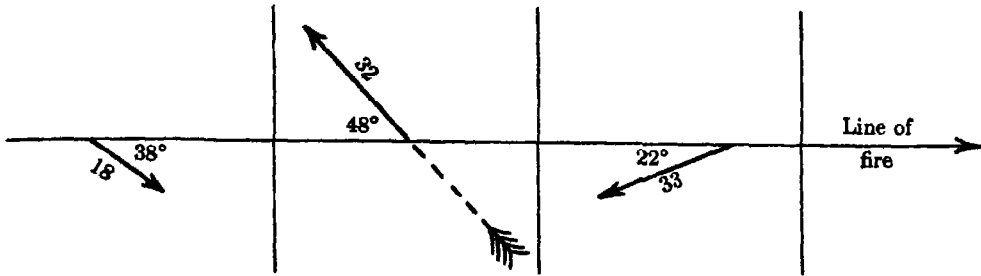
Since the relative bearing of the target and the target angle are given, we have the angles required for plotting the own-ship and target vectors (see examples (a) under articles 1226 and 1227). For the purpose of plotting the wind vector we find the wind angle as follows (see example (a) under article 1228).

Relative bearing of wind	=	10°
Relative bearing of target	=	322°
Reverse of relative bearing of wind	=	190°
Wind angle	=	132°

The diagram is shown above the form on the opposite page, and it shows that the errors due to own-ship motion are *over* in range and *right* in deflection, the errors due to target motion are *over* in range and *left* in deflection, and the errors due to wind are *short* in range and *left* in deflection.

Since the ballistic density is given, the multiplier for Column 12 is obtained, in this case, according to rule number II as stated in article 1212, i.e., by entering Table V with the given ballistic density and zero maximum ordinate.

Since apparent wind is to be used, rule number II as stated in article 1219 applies in this case, i.e., errors due to wind are to be found from Columns 13 and 16, and errors due to own-ship motion and target motion are in both cases to be found from Columns 15 and 18.



5''/3150 f.s. Gun

Target Distance 8500 yds.

				Range Errors		Deflection Errors	
				Over	Short	Left	Right
Col. 6 42.6						×	43
Col. 10 32	Erosion Powder temp. 79°F. (-) 11 × 2 = Total vel. var.	(-) 44 f.s. (-) 22 f.s. (-) 66 f.s.	66 — × 32 = 211 10	×	211		
Col. 11 (-) 5	Var. from standard weight of projectile	(-) 2 lbs.	2 × 5 = 10	×	10		
Col. 12 390	Ballistic density Multiplier (T.V)	.933 (+) .67	.67 × 390 = 261	261	×		
Col. 13 46	Wind velocity (apparent) ✕ with line of fire 32 cos 48° = 21.4	32 48°	21.4 — × 46 = 98 10	×	98		
Col. 15 74	Own ship's speed ✕ with line of fire 18 cos 38° = 14.2	18 38°	14.2 — × 74 = 105 10	105	×		
Col. 15 74	Target's speed ✕ with line of fire 33 cos 22° = 30.6	33 22°	30.6 — × 74 = 226 10	226	×		
Col. 16 27	Wind velocity (apparent) ✕ with line of fire 32 sin 48° = 23.8	32 48°	23.8 — × 27 = 64 10			64	×
Col. 18 74	Own ship's speed ✕ with line of fire 18 sin 38° = 11.1	18 38°	11.1 — × 74 = 82 10			×	82
Col. 18 74	Target's speed ✕ with line of fire 33 sin 22° = 12.4	33 22°	12.4 — × 74 = 92 10			92	×
Total range error..... 273 yds. Over				592	319	156	125
Total deflection error..... 31 yds. Left				319		125	
1 mil = $\frac{8500}{1000} = 8.5$ yds., $\frac{31}{8.5} = 3.6$				273		31	

Ballistic Correction in Range..... Down 273 yds.
 Ballistic Correction in Deflection..... Right 4 mils.
 Sight-bar Range 8,250 Scale 104

1233. Certain other corrections, in addition to those which have been considered in connection with the problems just solved, sometimes enter into the final value of the ballistic corrections which determine the sight-bar range and scale. The final value of the ballistic corrections is called the *total ballistic*, and it includes the *gun ballistic*, *control ballistic*, *arbitrary ballistic*, and *cold-gun correction*. The purpose and composition of each of these portions of the total ballistic is explained below. (See also Sheet 10.)

1234. The *gun ballistic* includes the corrections for drift, for variations from standard initial velocity, standard air density, and standard weight of projectile, and for wind, motion of gun, and motion of target, that is, all of the elements which have been included in the problems of articles 1231 and 1232, whence the ballistic corrections found in these problems are, in fact, the gun ballistic in range and in deflection.

1235. The *control ballistic* is designed to account for errors due to certain artificialities of control, such as the use of a point of aim which does not agree with the desired point of impact, or of a range scale which does not agree with the latest range table in effect. It includes the *point of aim correction*, and the *sight-scale-range-table correction*.* The point of aim correction has already been explained sufficiently in article 1221.

It sometimes happens that a gun is fitted with an obsolete range scale, although an up to date range table is available. The *sight-scale-range-table correction* is designed to bring the sight-bar range, as set on an obsolete range scale, into accord with the latest range table in effect, and if such a correction is required it is included in the control ballistic. An example will serve to illustrate how this correction is determined. Let us suppose that we have a gun for which a new range table has been issued, but that corresponding sight scales have not been provided. The ballistic corrections should then be computed as usual, using the new range table. Let us suppose that for a target distance of 20,000 yards we have found, according to a new range table, a gun ballistic in range of Up 500 yards, whence the sight-bar range would be 20,500 yards if the range scale were graduated according to the new range table. If the range scale is in fact graduated according to an obsolete range table, we must then adjust the sight-bar range as follows. Let us suppose now that in the new table the range 20,500 yards corresponds to the angle of elevation $12^{\circ} 41' 6''$, and that in the *old table* the angle of elevation $12^{\circ} 41' 6''$ corresponds to the range 20,800 yards. Then in order to give the gun the angle of elevation that is actually required, we must set on its range scale (which is graduated according to the old table) the sight-bar range 20,800 yards. The latter evidently can be considered to be the sight-bar range 20,500 yards, as determined according to the new range table, plus an additional correction of Up 300 yards. This correction of Up 300 yards is the sight-scale-range-table correction in this case, and it is added to the control ballistic. The total ballistic (assuming that no further corrections enter into this case) then becomes $500 + 300 = 800$ yards, whence the sight-bar range becomes $20,000 + 800 = 20,800$ yards, as required. In director fire a similar correction is

* On Sheet 10, vertical and horizontal parallax corrections are also included in the control ballistic. These corrections apply only in director fire, and in all modern director systems they are taken care of mechanically by the director system itself and do not enter into the ballistic corrections at all.

required if the range converter is not graduated according to the latest range table in effect.

The arbitrary ballistic 1236. The *arbitrary ballistic* is designed to account for errors which have been observed at previous firings but to which no specific cause can be assigned. It is determined by deducing from the actual results of a firing, the ballistic corrections that would have given the correct range and deflection, and by comparing these with the corrections that can be accounted for according to the best data available after the firing. In making such an analysis the proper procedure is to re-compute the ballistic corrections after the firing, and to compare them with the corrections indicated by actual results to have been necessary. The computations made *before* the firing may have been made on the basis of predictions that did not actually materialize. The arbitrary ballistic should not include errors in the pre-firing determination of the ballistic corrections that can be accounted for by post-firing analysis, such as erroneous estimates of target course or speed, wind, target distance, etc.

The best of judgment must be exercised in determining an arbitrary ballistic, and great care must be taken to exclude from it corrections for mistakes that occurred on a previous firing, for evidently the inclusion of such corrections in the arbitrary ballistic would serve merely to subject a subsequent firing to the effects of these mistakes. An arbitrary ballistic should be based on the records of as many previous firings as are available, but it must be borne in mind that re-lining of the guns, or the use of a new range table, will very likely invalidate any previously determined arbitrary ballistic. It is also worthy of note that the use of a new lot of powder may change the arbitrary ballistic materially, for unknown differences among various lots of powder designed for the same gun probably contribute as much as any other cause to the arbitrary ballistic.*

The cold-gun correction 1237. The *cold-gun correction* is designed to account for the reduction in range that usually results from the firing of cold guns; it is to be noted, in this connection, that the range table is based on the performance of the gun after a warming round has been fired (art. 803 (e)), and that the results obtained with a cold round are consequently very likely to differ from those indicated in the range table. Although cold-gun corrections are commonly applied only to the range, there is ample evidence that such corrections are required in deflection also. Cold-gun corrections can be determined only from careful post-firing analysis; they may change materially during the life of the gun, or upon re-lining. The causes and characteristics of cold-gun errors will be discussed further in Chapter 13.

Mechanical devices for handling ballistic corrections 1238. In actual practice, the manner in which the ballistic corrections are determined depends upon the equipment available. Most batteries are now provided with range-keepers which automatically supply some portions of the ballistic corrections. Those portions of the ballistic corrections which are not handled automatically by the range-keeper must be determined separately, and applied manually to the range-keeper. It is of great importance to understand clearly not only what corrections are handled automatically by the range-keeper, but also the manner in which the latter handles them. For example, many range-keepers generate the corrections for own-ship motion on the basis of Columns 15 and 18, and hence render it

* See also arts. 1906-1908.

necessary to use apparent wind in connection with the wind corrections (art. 1219). Director systems include *erosion correctors* which, when set up with the range and the rounds fired (or the reduced velocity corresponding to the latter), automatically adjust the gun elevation for erosion losses. If the erosion losses are thus accounted for, they must, of course, not be included in the ballistic corrections which are to be applied elsewhere. If, on the other hand, the erosion losses have been included in the ballistic corrections which are to be applied to the range-keeper, the erosion correctors must be set for zero rounds fired, or for zero range. In short, it is necessary to make a complete survey of the fire-control equipment in order to determine how the various elements of the ballistic corrections are to be handled.

1239. *Ballistic-computation diagrams* are provided for the determination of some portions of the ballistic corrections which are not handled automatically by the range-keeper or other mechanical devices. These diagrams are usually constructed in the form of *alignement charts*, i.e., combinations of several graphs in an arrangement which provides for the solution, by a purely graphical process, of quantities which depend on several arguments. For example, the range correction corresponding to a given set of surface atmospheric conditions depends upon the surface temperature and barometric pressure, and upon the maximum ordinate and value of Column 12 corresponding to the given range. The ballistic-computation diagram for this range correction is arranged so that when entered with temperature, barometer, and range, the operations of determining the surface density from Table III, the maximum ordinate from Column 8, the ballistic density and corresponding multiplier from Table V, and the range correction from Column 12, are all accomplished entirely graphically and in a few steps. The diagrams for wind corrections include the operations both of resolving the wind into its components in and across the line of fire and of applying the components to Columns 13 and 16. Detailed instructions concerning their purpose and use are printed on the ballistic-computation diagrams themselves.*

1240. Reduced-charge range tables (art. 1006) are provided for use in connection with reduced-charge firing, but the corresponding range scales are not always provided. It is a simple matter, however, to construct a reduced-charge range scale when both the full-charge and the reduced-charge range tables are available. An example will make the process clear. Let us suppose that a 16" gun is to be fired at target practice, at 1600 yards, with reduced charges which are designed to give an initial velocity of 2000 f.s., and that the gun is provided only with a full-charge range scale which is graduated for the gun's service velocity of 2600 f.s. We find that in the reduced-charge range table (16"2000 f.s.) the range 1600 yards corresponds to the angle of elevation $1^{\circ}08'5$, and that in the full-charge range table (16"2600 f.s.) the angle of elevation $1^{\circ}08'5$ corresponds to the range 2656 yards. Hence if we use the full-charge range scale and set it at 2656, we will have the correct setting for 1600 yards with the reduced charge. Or we can accomplish the same result by pasting

* For explanation to the construction of *alignement charts*, see pp. 179-186, *Marks' Mechanical Engineers' Handbook* (Second Edition).

over the full-charge scale a temporary reduced-charge scale so laid off that the graduation 1600 on the temporary scale corresponds to the graduation 2656 on the full-charge scale. Other graduations for the temporary reduced-charge scale can be determined similarly. Example:

Given: The 16" gun and its full-charge range table (16"2600 f.s.) and reduced-charge range table (16"2000 f.s.).

Find: The range graduations on the full-charge range scale to which the range graduations for 1500, 1600, 1700, and 1800 yards on the reduced-charge range scale must correspond.

By entering the 16"2000 f.s. range table with each of the given ranges and finding the corresponding angles of elevation, and then entering the 16"2600 f.s. range table with these angles of elevation and finding the corresponding ranges, we obtain the following relation between the graduations of the two scales:

Graduations of reduced-charge scale	Angle of elevation	Graduations of full-charge scale
yds.	° ' "	yds.
1500	1 04.1	2493
1600	1 08.5	2656
1700	1 13.0	2822
1800	1 17.5	2989

1241. A problem very much similar to the above arises in the use of sub-caliber guns. A sub-caliber gun is a small gun which is mounted rigidly upon a larger gun, and which is used for training purposes to simulate the firing of the larger gun. The latter is elevated and trained by means of its regular operating gear and sights. If the sub-caliber gun requires an angle of elevation of 5° for a given range, the sights of the large gun must be set at the range which corresponds to the angle of elevation 5° according to the large gun's range table. The 1-pounder gun is usually used for sub-caliber practices, and a 1-pounder range table is available for controlling its fire. For example:

Given: A 1-pounder gun is mounted upon the chase of a 16"2600 f.s. gun, and the bores of the two guns are parallel.

Find: The settings on the 16" gun's full-charge range scale that are required to give the ranges 500, 600, 700, 800, 900, and 1000 yards with the sub-caliber gun.

The required angles of elevation are found from the 1-pounder range table, and the ranges corresponding to these angles in the 16"2600 f.s. range table are the required settings for the 16" gun's full-charge range scale.

1-pdr. range	Angle of elevation	16" full-charge range scale setting
yds.	° ' "	yds.
500	0 27.5	1100
600	0 35.1	1392
700	0 43.6	1719
800	0 53.0	2081
900	1 03.3	2463
1000	1 14.7	2885

1242. If the sub-caliber gun's range is controlled by means of the large gun's range scale, the spotter's corrections evidently must be converted to the same scale before being applied to it. It can be seen from the tabulation given above that

in order to increase the 1-pounder gun's range from 500 yards to 600 yards, the 16" full-charge range scale setting must be increased from 1100 yards to 1392 yards; that is, an increase of only 100 yards in the actual range requires, in this case, an increase of 292 yards in the sight-bar range. Similarly, a 100-yard increase in the actual range from 900 yards to 1000 yards requires a 422-yard increase in the sight-bar range (i.e., from 2463 yards to 2885 yards). This means, then, that for actual ranges around 500 yards, the spotter's corrections must be multiplied by about 3 before being applied, while for actual ranges around 1000 yards they must be multiplied by about 4 before being applied. The necessity for such a highly artificial procedure can be avoided by preparing a sub-caliber range scale and pasting it over the large gun's range scale.

1243. Another method that may be employed very conveniently in connection with main battery guns, is the use of a conversion scale on the range ruler of the Mark II plotting board. This method serves equally well for the reduced-charge problem and for the sub-caliber problem. It is identical in principle with the method of substituting for the regular full-charge scale a reduced-charge or sub-caliber scale, but has the advantage of requiring the preparation of but one scale, instead of one for each gun, and of eliminating the undesirable feature of smearing the full-charge scales.

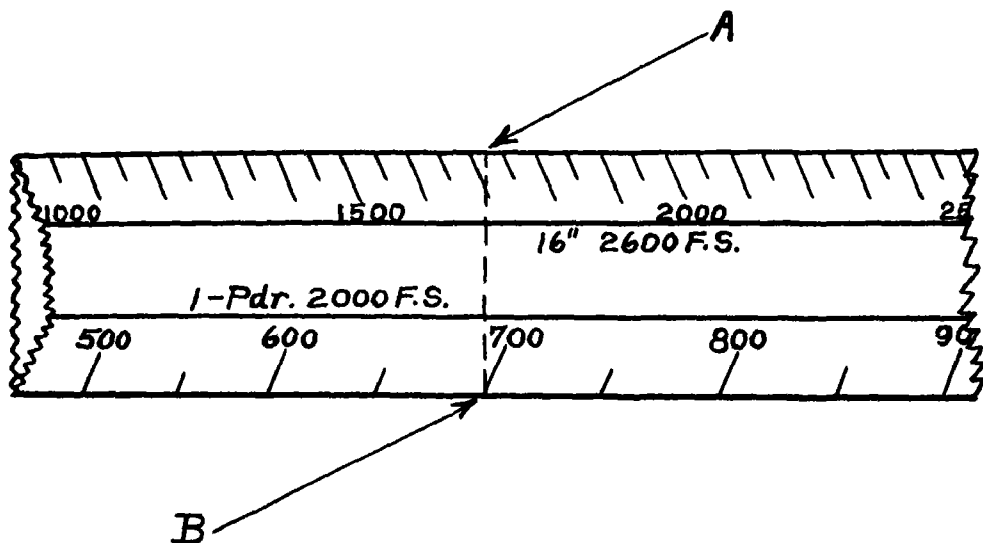


FIGURE 36

Figure 36 represents a section of the range ruler of the Mark II board. Scale A is the usual scale engraved on the upper edge of the ruler; it is labeled, for the purpose of this illustration, 16" 2600 f.s. Scale B represents a temporary scale, marked on paper or some other suitable material and pasted along the bottom edge of the ruler; it is labeled, for the purpose of this illustration, 1-Pdr. 2000 f.s.

The graduations of scale B have been made to scale in accordance with the figures of the problem solved in article 1241. For instance, the 700-yard graduation of scale B was located by placing it vertically beneath the point representing 1719 yards on scale A (for the example shows that the 700-yard sub-caliber range corresponds to the 1719-yard range on the 16" 2600 f.s. scale). Other points were located similarly.

Having graduated the ruler in this manner, range-finder ranges, ballistic corrections, and spots can now be plotted against scale B, and sight-bar ranges

read from scale A. Various other arrangements based on the principle illustrated in Figure 36 may also be devised.

EXERCISES

1. *Given:* The target distance and the total ballistic.

Find: The sight-bar range and scale. (Nearest 50 yds. and whole mil.)

	Given			Answers	
	Target distance (yds.)	Total ballistic		Sight-bar range (yds.)	Scale (mils)
		Range (yds.)	Deflection (yds.)		
A	2000	Up 163	Right 12	2150	106
B	8500	Down 348	Right 22	8150	103
C	14,000	Down 412	Left 40	13,600	97
D	19,300	Up 580	Right 152	19,900	108
E	32,000	Up 874	Left 620	32,850	81

2. *Given:* The type of gun, the powder temperature, and the velocity loss due to erosion.

Find: From the range table, at the given range, the range error due to the variations from the standard initial velocity.

	Given				Answers
	Gun	Range (yds.)	Powder temp. (°F.)	Erosion loss (f.s.)	Range error (yds.)
A	3"2700 f.s.	3000	85	6	(-) 22
B	4"2900 f.s.	1600	82	16	(-) 32
C	5"3150 f.s.	12,000	78	14	(-) 148
D	12"2900 f.s.	20,000	80	32	(-) 472
E	16"2600 f.s.	32,000	76	50	(-) 1443

3. *Given:* The type of gun, the surface air temperature, and the surface barometric pressure.

Find: From the range table, at the given range, the range error for the ballistic density which is assumed to correspond to the given surface conditions.

	Given				Answers	
	Gun	Range (yds.)	Surface conditions		Multiplier, Table V	Range error (yds.)
			Air temp. (°F.)	Barometer (inches)		
A	3"2700 f.s.	3000	52	30.25	(-) .40	(-) 49
B	4"2900 f.s.	1600	62	30.30	(-) .20	(-) 4
C	5"3150 f.s.	12,000	91	29.20	(+) .69	(+) 444
D	12"2900 f.s.	20,000	77	29.80	(+) .20	(+) 148
E	16"2600 f.s.	32,000	88	29.12	(+) .28	(+) 358

4. *Given:* The type of gun, and the ballistic density as determined for the given range by the aerological party.

Find: From the range table, at the given range, the range error corresponding to the given ballistic density.

	Given			Answers	
	Gun	Range (yds.)	Ballistic density	Multiplier	Range error (yds.)
A	3"2700 f.s.	3000	1.040	(-).40	(-) 49
B	4"2900 f.s.	1600	1.020	(-).20	(-) 4
C	5"3150 f.s.	12,000	.940	(+).60	(+) 386
D	12"2900 f.s.	20,000	.982	(+).18	(+) 133
E	16"2600 f.s.	32,000	.963	(+).37	(+) 474

5. *Given:*

16"2600 f.s. gun
 Target distance = 33,100 yards
 Surface air temperature = 84°F.
 Surface barometer = 29".30
 Magazine temperature = 78°F.
 Erosion loss = 58 f.s.

Relative bearing of target = 355°
 Own ship's speed = 20 knots
 Target angle = 345°
 Target's speed = 18 knots
 Velocity of ballistic apparent wind = 35 knots
 Relative bearing of ballistic apparent wind = 0°

Find:

Gun ballistic in range and deflection.
 Sight-bar range and scale.

Answers:

Up 615 yards, Left 33 mils
 Range, 33,700, Scale 67

6. *Given:*

14"2600 f.s. gun
 Target distance = 14,100 yards
 Ballistic density = .974
 Magazine temperature = 80°F.
 Erosion loss = 42 f.s.

Own ship's true course = 150°
 Own ship's speed = 16 knots
 True bearing of target = 65°
 Target's true course = 190°
 Target's speed = 14 knots
 Velocity of ballistic apparent wind = 24 knots
 Relative bearing of ballistic apparent wind = 330°

Find:

Gun ballistic in range and deflection.
 Sight-bar range and scale.

Answers:

Up 336 yards, Left 6 mils
 Range 14,450, Scale 94

7. *Given:*

16"2600 f.s. gun
 Target distance = 22,700 yards
 Surface air temperature = 64°F.
 Surface barometer = 29".60
 Magazine temperature = 78°F.
 Erosion loss = 28 f.s.

Own ship's true course = 70°
 Own ship's speed = 20 knots
 True bearing of target = 355°
 Target's true course = 90°
 Target's speed = 22 knots
 Velocity of ballistic true wind = 16 knots
 True bearing of ballistic true wind = 45°

Find:

Gun ballistic in range and deflection.
Sight-bar range and scale.

Answers:

Up 657 yards, Left 3 mils
Range 23,350, Scale 97

8. *Given:*

14"2600 f.s. gun
Target distance = 14,000 yards
Ballistic density = .990
Magazine temperature = 82°F.
Erosion loss = 38 f.s.

Relative bearing of target = 265°
Own ship's speed = 16 knots
Target angle = 55°
Target's speed = 12 knots
Velocity of ballistic apparent wind = 26 knots
Relative bearing of ballistic apparent wind = 340°

Find:

Gun ballistic in range and deflection.
Sight-bar range and scale.

Answers:

Up 346 yards, Left 6 mils
Range 14,350, Scale 94

9. *Given:*

16"2600 f.s. gun
Target distance = 24,700 yards
Surface air temperature = 64°F.
Surface barometer = 30".27
Magazine temperature = 74°F.
Erosion loss = 20 f.s.

Own ship's true course = 50°
Own ship's speed = 16 knots
Target's true bearing = 330°
Target's true course = 80°
Target's speed = 18 knots
Velocity of ballistic true wind = 14 knots
True bearing of ballistic true wind = 30°

Find:

Gun ballistic in range and deflection.
Sight-bar range and scale.

Answers:

Up 773 yards, Left 7 mils
Range 25,450, Scale 93

10. *Given:* The type of gun and a sight-bar range for that gun.

Find: The corresponding range of the 1-pdr. gun, assuming that the latter is rigidly mounted upon the large gun and that its bore is parallel with that of the large gun.

	Given		Answers
	Gun	Large Gun S. B. Range	1-Pdr. Range
A	3"2700 f.s.	3000	1356
B	4"2900 f.s.	1100	448
C	5"3150 f.s.	1700	559
D	12"2900 f.s.	20400	3408
E	16"2600 f.s.	2600	932

CHAPTER 13

THE EFFECTS OF THE EARTH'S ROTATION ON THE TRAJECTORY. ERRORS DUE TO TRUNNION TILT. COLD-GUN ERRORS.

New Symbols Introduced.

- ΔX_0 Change in range due to earth's rotation.
 D_0 Lateral deviation due to earth's rotation.
 A, B, D Rotation coefficients (see art. 1305); the rotation coefficient D is not to be confused with the drift, which is represented by the same symbol.
 α Azimuth of the line of fire, measured clockwise from north.
 l Geographical latitude.
 t The angle of trunnion tilt, i.e., the angle at which the axis of the gun trunnions is inclined with respect to the horizontal.
 ϕ_t' The effective angle of elevation as modified from ϕ' by trunnion tilt.
 t_D The angular deflection resulting from trunnion tilt.
 D_t Lateral deviation due to trunnion tilt, i.e., linear deflection corresponding to the angular deflection t_D .

1301. The effects of the earth's rotation on the trajectory vary according to the latitude of the gun's location and the azimuth of the line of fire. The assumption of a motionless earth for range-table values (art. 801 (g)) therefore belongs in the same category as the assumption of zero values for other forces which may occur in various combinations of amount and direction. That is to say, just as range-table values may be adjusted for the effects of wind, motion of gun, and motion of target, according to the amount and direction of each of these forces, they may be adjusted also for the effects of the earth's rotation according to the latitude of the gun's location and the azimuth of the line of fire. The errors in range and deflection due to the earth's rotation are not listed in the range table, but they can be determined from formulas which are given in this chapter.*

For the purposes of logical explanation, the effects of the earth's rotation may be resolved into three components which we shall call, for convenience in referring to them, the X -component, the Y -component, and the Z -component. The X -component affects only the range, the Y -component both the range and the deflection, and the Z -component only the deflection. We shall now examine these three components in turn.

1302. Let us suppose that a gun is located at the equator and fired due east. As a result of the earth's rotation, points on the equator are carried through space to the eastward, along a circular path, at a linear velocity of about 1500 f.s.† The projectile, however, after it has left the gun, retains this velocity of 1500 f.s. in a fixed direction with respect to space,

* There appears to be no good reason for not listing these errors in the range tables of the U. S. Navy; they have been listed in the firing tables (range tables) of the U. S. Army since about 1920.

† This is found by dividing the earth's circumference at the equator by the number of seconds in a sidereal day.

i.e., in the direction occupied at the instant of projection by the tangent to the equator at the position of the gun. Let us suppose that, under the range-table assumption of a motionless earth, the point of fall would occur at a distance of 30,000 yards from the gun. The situation then is that the projectile (apart from the motion imparted to it by the charge of the gun) and the predicted point of fall are both in motion to the eastward at a linear velocity of about 1500 f.s., but that the projectile's motion is along a straight path, while the motion of the predicted point of fall is along a circular path. This situation is illustrated in Figure 37, in which EE' represents the earth's surface at the equator, OSH a trajectory based on the assumption of a motionless earth, and HH_1 the distance through which the earth's rotation has carried the predicted point of fall (H) during the predicted time of flight. The projectile, as a result of the earth's rotation, has acquired motion in the direction of the tangent OO' . In the predicted time of flight this motion will have caused displacement of the projectile from the pre-

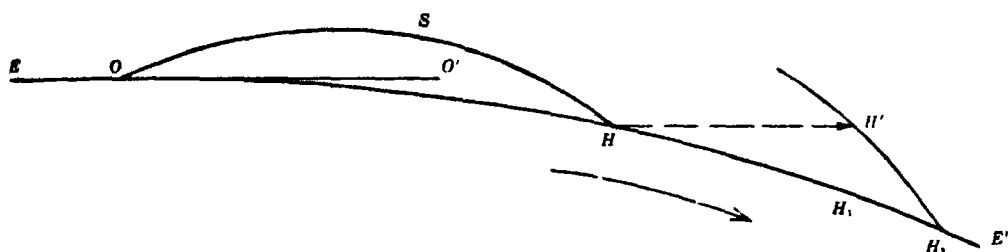


FIGURE 37

dicted position H to the position H' , the displacement HH' being equal in amount to HH_1 but being parallel in direction to OO' . At the expiration of the predicted time of flight the projectile will then still be above the earth's surface, and the actual point of fall will occur at H_2 , thus causing the range to be increased by the amount H_1H_2 . In this case (firing to eastward) there is an apparent falling away of the earth's surface from the projectile, as a consequence of which the latter's flight is prolonged and the range increased. It can be shown similarly that in the case of firing to westward there is an apparent rising of the earth's surface with respect to the projectile, as a consequence of which the latter returns sooner to the earth and the range is decreased.

The effect of the X -component varies in magnitude according to the linear velocity of the earth's surface with respect to the direction of fire. The linear velocity itself varies as the cosine of the latitude, or as $\cos l$; it is greatest at the equator and vanishes at the poles. At a given latitude, the linear velocity of the earth's surface with respect to the direction of fire varies as the sine of the azimuth of the line of fire, or as $\sin a$ (a being measured clockwise from the north); it is greatest for fire to the east or west, and vanishes for fire to the north or south. It is to be noted that the X -component does not directly affect the deflection.

1303. Let us now suppose that a projectile is fired vertically from a position on the equator. In order to remain vertically* above the point on the earth's surface from which it was fired, the projectile, as it rises, evidently
 The Y -component would have to increase its velocity to the eastward in proportion to its distance from the center of the earth. Actually, however, the

* The term *vertical* as used in this discussion is to be interpreted as defining a direction perpendicular to the earth's surface.

projectile's eastward velocity is limited to that of the point on the earth's surface from which the projectile was fired. As the projectile ascends it therefore lags, by an ever-increasing amount, to the westward of a line extending vertically upward from the gun, and at the end of its ascent the projectile will be vertically above a point on the earth's surface which is to the westward of the gun. At the end of its ascent the projectile will have lost all of the motion imparted to it by the charge of the gun, but will have retained the eastward velocity imparted to it by the earth's rotation. Its motion in descent will be the resultant of this eastward velocity and of the gravitational pull vertically downward. We have seen that at the end of its ascent the projectile is vertically above a point to the westward of the gun; for convenience, let us refer to this point to the westward of the gun as point *B*. At the beginning of the projectile's descent the gravitational pull is then vertically downward toward point *B*. But as the projectile, in its descent, lags to the westward of the vertical with respect to point *B*, the gravitational pull will, accordingly, shift progressively toward points on the earth's surface lying to the westward of point *B*. The projectile then will eventually return to the earth at a point somewhat to the westward of point *B*, and hence to the westward of the gun.

For other than vertical fire, points on the actual trajectory will similarly lag to the westward of the corresponding points on the trajectory calculated under the assumption of a motionless earth. As a result of this lag to the westward, the range will be decreased if the direction of fire is to the eastward, and increased if the direction of fire is to the westward. It will be observed that the effects of the *Y*-component on the range are thus opposite in direction to those of the *X*-component (art. 1302).

The effect of the *Y*-component varies in magnitude according to the linear velocity of the earth's surface with respect to the direction of fire. The linear velocity itself varies as the cosine of the latitude, or as $\cos l$; it is greatest at the equator and vanishes at the poles. In the case of fire to the eastward or westward, the westward lag caused by the *Y*-component operates wholly in the line of fire and hence affects only the *range*; in the case of fire to the northward or southward, the lag is still to the westward, and hence it operates wholly *across* the line of fire and affects only the *deflection*. For intermediate directions of fire the westward lag has a component *in* the line of fire which affects the *range*, and a component *across* the line of fire which affects the *deflection*. The range component varies as $\sin a$ and the deflection component as $\cos a$ (where *a* is the azimuth of the line of fire, measured clockwise from the north). Since the lag is always to the westward, the deflection errors resulting therefrom are to the *left* when the direction of fire is in the *northerly* quadrants (i.e., between 0° and 90° , or between 270° and 360°), and to the *right* when the direction of fire is in the *southerly* quadrants (i.e., between 90° and 180° , or between 180° and 270°).

1304. The *Z*-component of the rotational effects results from the fact that the line from the gun to the target changes its direction constantly with respect to space (unless the gun is at the equator), whereas the line of fire moves parallel to itself through space and hence does not change its direction with respect to space. Figure 38 (a) represents the situation that obtains in the case of fire from or to one of the earth's poles. Let us assume first that the gun is exactly at the north pole, *P*, and the target at *A*, and that at the instant of fire the gun is trained in the direction *PA*. At the pole the linear velocity of the earth's surface is, of course, zero, and the projectile therefore acquires no motion from the earth's rotation but travels through space

in the direction PA . But the earth's rotation during the time of flight carries the target to A' , and the point of fall therefore occurs to the right of the target. The amount of the lateral deviation in this case is approximately equal to the product of the time of flight and the linear velocity of the earth's surface at A ; or, expressed in mils, it is approximately equal to the product of the time of flight and the angular velocity (in mils) of the earth's rotation.*

Let us suppose now that the gun is at A and the target at P (Figure 38 (a)), and that at the instant of fire the gun is trained in the direction AP . In this case the earth's rotation imparts no motion to the target, but imparts to the projectile motion in the direction AA'' , normal to AP , at the linear velocity of the earth's surface at A . The point of fall will therefore again occur to the *right* of the target by an amount approximately equal to the product of the time of flight and the linear velocity of the earth's surface at A . It is to be noted that the deviation has been to the right in both cases illustrated, although the direction of fire in the first case was to southward and in the second case to northward.

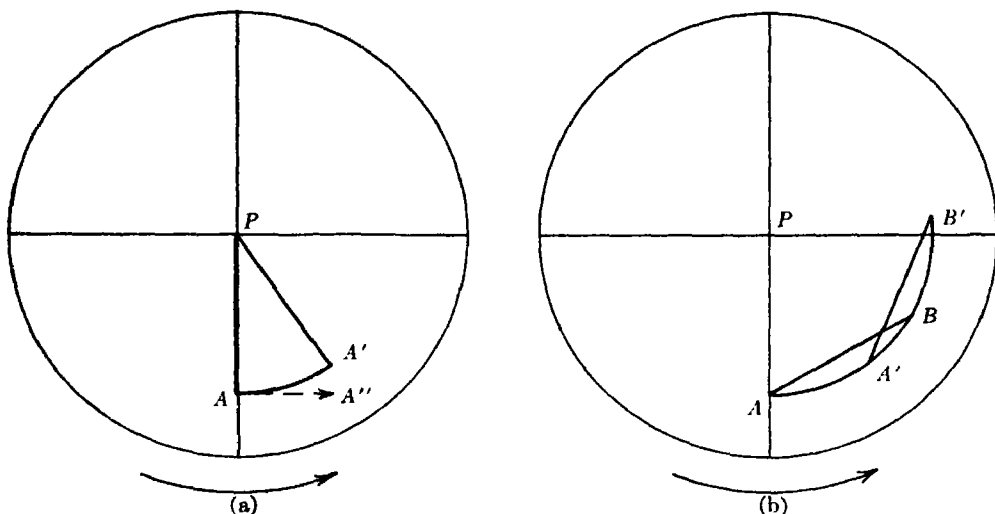


FIGURE 38

Figure 38 (b) represents the situation that obtains in the case of fire to eastward or westward near one of the earth's poles. Let us assume first that the gun is located at A , and the target at B (A and B being equidistant from the north pole, P), and that at the instant of fire the gun is trained in the direction AB . During the time of flight, however, the direction of the line joining the gun and target will have changed from AB to $A'B'$, and the projectile, having been fired in the initial direction AB , will consequently fall to the *right* of the target. Assuming that the gun and target are very near the pole, A , B , and P (Fig. 38 (b)) lie in practically a flat plane, and under this condition the change in direction between AB and $A'B'$ will be practically equal to the angle through which the earth has rotated during the time of flight. The case just considered represents the situation for fire to eastward. In order to examine the situation for fire to westward, let us assume that the gun is at B and the target at A . The line from gun to target changes its direction, in this case, from BA to $B'A'$ during the time of flight, and the initial direction BA which is imparted to the projectile will again cause the latter to fall to the *right* of the target.

*The angular velocity of the earth's rotation is equal to 2000π mils divided by the number of seconds in a sidereal day, or $6283.2/86,164 = .073$ mils per second.

We have now considered the cases of fire to the northward, southward, eastward, and westward (near the *north* pole), and have found in each case that the lateral deviation is to the *right*. We have also found that the change in direction of the line from gun to target, which is responsible for this deviation, depends on the same factors *irrespective of the direction of fire*, being in all cases equal to the angle through which the earth rotates during the time of flight (which applies, however, only to points very near the pole, as here assumed).

Figure 39 illustrates how a change of latitude affects the situation. In this figure the sets of lines AB and $A'B'$, A_1B_1 and $A_1'B_1'$, A_2B_2 and $A_2'B_2'$, etc.,

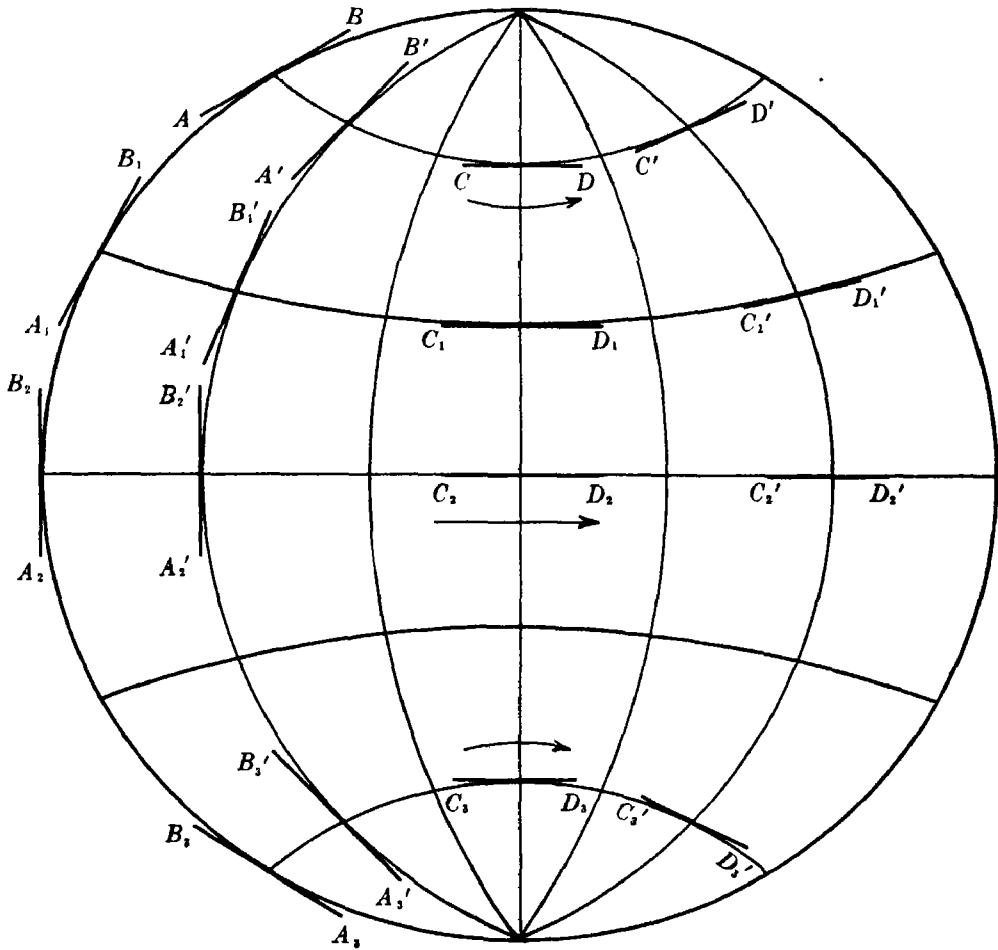


FIGURE 39

illustrate how the directions of north and south lines are changed, with respect to space, by the earth's rotation. It will be observed that the change in direction is greatest near the poles, as illustrated by the lines AB and $A'B'$. At the equator the change in direction due to rotation vanishes, the successive positions of the line A_2B_2 being parallel to each other. In the southern hemisphere we find that the changes in direction are such as to cause deviation to the *left*. (Note that the initial direction A_3B_3 inclines to the *left* of the final direction $A_3'B_3'$, and that the initial direction B_3A_3 also inclines to the *left* of the final direction $B_3'A_3'$; compare with the set A_1B_1 and $A_1'B_1'$.) The effect of the Z -component is therefore greatest at the poles and zero at the equator, and it varies as the sine of the lati-

tude, i.e., as $\sin l$, and is independent of the direction of fire. The sets of lines CD and $C'D'$, C_1D_1 and $C_1'D_1'$, etc., show similarly how the directions of east and west lines are changed, with respect to space, by the earth's rotation.

1305. Comparison between the differential equations of the trajectory referred to a set of axes rotating in the direction of and with the angular velocity of the earth's rotation, and the differential equations of the trajectory referred to a set of fixed axes (i.e., a motionless earth), shows that the terms representing the effects of rotation take the forms

$$\Delta X_0 = A \cos l \sin a \quad (1301)$$

$$D_0 = B \sin l - D \cos l \cos a \quad (1302)$$

in which ΔX_0 and D_0 are, respectively, the change in range and the lateral deviation due to rotation, l is the latitude, a is the azimuth of the line of fire (measured clockwise from the north), and A , B , and D are coefficients which depend on the trajectory and whose exact values can be obtained only by integration of certain terms in the differential equations. Sufficiently approximate values of A , B , and D may be found, however, from the expressions.*

$$A = 0.00002431 T (346 \Delta X_{\phi, 10} + 1.8 X \cot \omega) \quad (1303)$$

$$B = 0.00003646 TX \left(\frac{3 \tan \phi + \tan \omega}{\tan \phi + \tan \omega} \right) \quad (1304)$$

$$D = 0.00001215 TX \tan \phi \left(\frac{19 \tan \phi + \tan \omega}{7 \tan \phi + 3 \tan \omega} \right) \quad (1305)$$

in which X is the horizontal range, T the time of flight, ϕ the angle of departure, ω the angle of fall, and $\Delta X_{\phi, 10}$ is the change in range corresponding to an increase of 10 minutes in the angle of departure (which can be determined from Column 2 b of the range table).

1306. The form of (1301) indicates that the coefficient A is associated with both the X -component of rotation (art. 1302), and that portion of the Y -component which affects the range (art. 1303), for we have found that the range effects due to either of these components vary as $\cos l$ and $\sin a$. For any angles of departure which occur in naval surface fire, the effect of the X -component is greater than that of the Y -component (in range); ΔX_0 therefore takes the sign of the X -component's effect, which is positive (i.e., such as to increase the range) for fire to eastward, and negative (i.e., such as to decrease the range) for fire to westward. The term $\sin a$ duly takes care of these signs, since $\sin a$ is positive for easterly quadrants and negative for westerly quadrants ($\cos l$, of course, remains positive for any value of l in either hemisphere).

The form of (1302) indicates that the coefficient B is associated with the Z -component of rotation, which varies as $\sin l$ (art. 1304), and that the coefficient D is associated with that portion of the Y -component which affects the deflection and which varies as $\cos l \cos a$ (art. 1303). We have found that the lateral deviation due to the Z -component is positive (i.e., to the right) in the northern hemisphere, and negative (i.e., to the left) in the southern hemisphere; in order

* These expressions, except for a slight simplification in the expression for A , are in the form given by Dr. T. H. Gronwall in *T.S. 162* (April, 1921, U. S. War Department). For strictly mathematical treatments of the effects of rotation, see arts. 4 and 34, *New Methods in Exterior Ballistics*, F. R. Moulton; also, Chapter XIII, *A Course in Exterior Ballistics*, R. B. Hoar (U. S. War Department Document No. 1051, December, 1920).

that the term $B \sin l$ may have the proper sign, l must therefore be considered as a positive angle for north latitudes and as a negative angle for south latitudes. We have found that the lateral deviation due to the Y -component is positive for directions of fire lying in the quadrants 90° to 180° and 180° to 270° , and negative for directions of fire lying in the quadrants 0° to 90° and 270° to 360° . The minus sign before the term $D \cos l \cos a$, in conjunction with the sign of $\cos a$, duly takes care of the sign of the Y -component's effect. For any angles of departure which occur in naval surface fire, the value of the term $D \cos l \cos a$ remains small, and is exceeded by the value of the term $B \sin l$ for latitudes greater than about 10° . Consequently, except within about 10° of the equator, the sign of D_α is positive in the northern hemisphere, and negative in the southern hemisphere, irrespective of the direction of fire. That is to say, the lateral deviation due to the earth's rotation is always to the right in the northern hemisphere and always to the left in the southern hemisphere, except near the equator, and near the equator it is always small.

1307. The calculation of the effects of the earth's rotation is illustrated in the following example.

Calculation of rotation effects *Given:* The 16"2600 f.s. gun and its range table; range 30,000 yards; latitude 45° north; direction of fire 240° true.
Find: The change in range and the lateral deviation due to the earth's rotation.

From the 16"2600 f.s. range table we have the data

$$X = 30,000 \text{ yards}$$

$$T = 54.49 \text{ seconds (Col. 4)}$$

$$\phi = 23^\circ 19' 0'' \text{ (Col. 2)}$$

$$\omega = 34^\circ 04' \text{ (Col. 3)}$$

$$\Delta X_{\phi, \omega} = \frac{10}{8.2} \times 100 = 122 \text{ yards (Col. 2b)}$$

The calculation of the values of the coefficients A , B , and D follows.

$$\phi = 23^\circ 19' 0''$$

$$\omega = 34^\circ 04'$$

$$\tan \phi = 0.43101$$

$$\tan \omega = 0.67620$$

$$3 \tan \phi = 1.2930$$

$$3 \tan \omega = 2.0286$$

$$7 \tan \phi = 3.0171$$

$$19 \tan \phi = 8.1892$$

$\Delta X_{\phi, \omega} = 122$	log 2.08636	$X = 30,000$	log 4.47712
346	log 2.53908	$\omega = 34^\circ 04'$	lcot 0.16992
42,212	log 4.62544	1.8	log 0.25527
79,856				log 4.90231
122,068	log 5.08661			
$T = 54.49$	log 1.73632			
0.00002431	log 5.38578 - 10			
$A = 161.70$	log 2.20871			
0.00003646	log 5.56182 - 10			
$X = 30,000$	log 4.47712			
$T = 54.49$	log 1.73632			
$(3 \tan \phi + \tan \omega) = 1.9692$	log 0.29429			
$(\tan \phi + \tan \omega) = 1.1072$	colog 9.95577 - 10			
$B = 109.50$	log 2.03942			

0.00001215.....	log 5.08458-10
$X=30,000$	log 4.47712
$T=54.49$	log 1.73632
$\phi=23^{\circ}19'0$	$\tan 9.63449-10$
$(19 \tan \phi + \tan \omega)=8.8654$	log 0.94770
$(7 \tan \phi + 3 \tan \omega)=5.0457$	colog 9.29708-10
$D=15.041$	log 1.17729

The calculation is now completed by the solution of (1301) and (1302).

$A=161.70$	log 2.20871	
$l=45^{\circ}$	$\cos 9.84949-10$	
$a=240^{\circ}(60^{\circ})$	$(-)\sin 9.93753-10$	
$\Delta X_0=(-)99$ yds.....	$(-)\log 1.99573$	
$B=109.50$	log 2.03942	$D=15.041$ log 1.17729
$l=45^{\circ}$	$\sin 9.84949-10$	$l=45^{\circ}$ $\cos 9.84949-10$
77.4.....	log 1.88891	$a=240^{\circ}(60^{\circ})$... $(-)\cos 9.69897-10$
		$(-)5.3$ $(-)\log 0.72575$
$D_0=77.4-(-)5.3=83$ yards		

The results obtained show that the earth's rotation in this case causes a reduction of 99 yards in the range, and a lateral deviation of 83 yards to the right.

1308. The present practice in the preparation of U. S. Navy range tables is to ignore the effects of the earth's rotation altogether. That is to say, the range and lateral deviation observed at the proving ground, in connection with the ranging of the gun for range-table data (art. 810), are not adjusted for the effects of the earth's rotation.* Consequently the range and drift, as tabulated in our range tables, include the effects of the earth's rotation for the latitude of the proving ground and for the direction of fire at which the ranging was conducted, and corrections to the range-table data must be adjusted accordingly. Practically all U. S. Navy range tables now in service are based on ranging conducted at the U. S. Naval Proving Ground at Dahlgren, Va., in latitude 38°N and in the direction 120° true. In order to determine, for a given latitude and direction of fire, rotation effects that are applicable to these range tables, it is therefore necessary to subtract from the ΔX_0 and D_0 for the given l and a , the ΔX_0 and D_0 for $l=38^{\circ}\text{N}$ and $a=120^{\circ}$. This operation can be incorporated in the rotation formulas themselves as follows, thus making the formulas directly applicable to the range tables and eliminating the necessity of making double computations.

Formulas for determination of rotation effects in connection with U. S. Navy range tables

$$\Delta X_0 = A(\cos l \sin a - 0.68244) \quad (1306)$$

$$D_0 = B(\sin l - 0.61566) - D(\cos l \cos a + 0.39400) \quad (1307)$$

The numerical terms that appear in (1306) and (1307) are, of course, merely the

* The practice of ignoring the rotation effects in connection with the proving-ground ranging can hardly be justified, especially in view of the fact that corrections are applied for other causes whose influence is often no greater than that of the earth's rotation. It approaches the absurd, for example, to apply to the observed range a correction for the height of the gun and to ignore the correction for rotation, for at long ranges the height-of-gun correction is generally but a small fraction of the rotation correction. The rotation corrections may, indeed, often be comparable in magnitude to any of the corrections actually applied to the observed range in connection with the proving-ground ranging.

required combinations of the functions for $l=38^\circ$ and $a=120^\circ$. For example, the operation

$$D \cos l \cos a - D \cos 38^\circ \cos 120^\circ$$

results in the expression

$$D \cos l \cos a - D(0.78801)(-0.50000)$$

or

$$D(\cos l \cos a + 0.39400)$$

which appears in (1307), and the other numerical terms were determined similarly.

1309. The following example illustrates the application of formulas (1306) and (1307).

Calculation of rotation effects in connection with U. S. Navy range tables *Given:* The 16"2600 f.s. gun and its range table; range 30,000 yards, latitude 45° north; direction of fire 240° true.
Find: The change in range and the lateral deviation due to the earth's rotation, considering that the range table includes these effects for the latitude and direction of fire at which the proving-ground ranging was conducted.

The coefficients A , B , and D are the same as already found in the example given in article 1307. Using these in connection with formulas (1306) and (1307), we then have

$l=45^\circ$	lcos 9.84949-10
$a=240^\circ (60^\circ)$	(-)lsin 9.93753-10
(-)0.61238.....	(-)log 9.78702-10
(-)0.68244.....	
(-)1.29482.....	(-)log 0.11220
$A=161.70$	log 2.20871
$\Delta X_D=(-)209$ yards.....	(-)log 2.32091

$B=109.50$log 2.03942	$l=45^\circ$lcos 9.84949-10
$l=45^\circ$..sin0.70711	$a=240^\circ (60^\circ)$..(-)lcos 9.69897-10
(-)0.61566	(-)0.35356..(-)log 9.54846-10
0.09145 log 8.96118-10	(+)0.39400
10.0.....log 1.00060	0.04044...log 8.60681-10
	$D=15.041$...log 1.17729
	0.6.....log 9.78410-10

$$D_D=10.0-0.6=9 \text{ yards}$$

The results obtained show that the earth's rotation in this case causes a reduction of 209 yards in the range, and a lateral deviation of 9 yards to the right, with respect to the range-table values.

1310. The following example illustrates how the signs are to be used in connection with the solution of (1306) and (1307) for a problem involving south latitude. Assuming the same data as already stated for the example in article 1309, except that the latitude is 45° south, the solution is then as follows.

$l = (-)45^\circ$	$\text{loos } 9.84949 - 10$
$a = 240^\circ (60^\circ)$	$(-) \text{lsin } 9.93753 - 10$
$(-)0.61238$	$(-) \text{log } 9.78702 - 10$
$(-)0.68244$	
$(-)1.29482$	$(-) \text{log } 0.11220$
$A = 161.70$	$\text{log } 2.20871$
$\Delta X_a = (-)209 \text{ yards}$	$(-) \text{log } 2.32091$

$B = 109.50$	$\text{log } 2.03942$	$l = (-)45^\circ$	$\text{loos } 9.84949 - 10$
$l = (-)45^\circ (-) \sin 0.70711$		$a = 240^\circ (60^\circ) (-) \text{loos } 9.69897 - 10$	
$(-) 0.61566$		$(-)0.35356 (-) \text{log } 9.54846 - 10$	
$(-) 1.32277$	$\text{log } 0.12149$	$(+)0.39400$	
$(-)144.8$	$(-) \text{log } 2.16091$	$0.04044 \dots \text{log } 8.60681 - 10$	
		$D = 15.041 \dots \text{log } 1.17729$	
		$0.6 \dots \text{log } 9.78410 - 10$	

$D_a = (-)144.8 - 0.6 = (-)145 \text{ yards}$

The results obtained show that the earth's rotation in this case causes a reduction of 209 yards in the range, and a lateral deviation of 145 yards to the left, with respect to the range-table values. It will be observed that the change from north to south latitude has affected only the term $B \sin l$; ΔX_a therefore has not been affected at all, although D_a has changed very materially.

ERRORS DUE TO TRUNNION TILT

1311. It is common knowledge to those familiar with the use of small arms that a rifle must be held so that its sights are vertical in order to shoot correctly, and that if the rifle is canted, the bullet is deflected to the side toward which the sights are inclined. What is commonly termed *cant* in the case of a rifle, is termed *trunnion tilt* in the case of a gun. The trunnions of a gun are said to be tilted when the axis of the trunnions is not horizontal, and the angle of trunnion tilt, which we shall denote by t , is the angle between the horizontal plane and the axis of the trunnions. Trunnion tilt operates to reduce the range and to cause lateral deviation toward the side of the lower trunnion. The reason for this will be apparent if the extreme case is considered. Let us suppose that the range scale of a gun has been set for a range corresponding to the angle of elevation 5° , and that the deflection scale has been set at 100 (i.e., so that the line of sight is parallel to the bore). If the gun is now tilted through 90° , so that its trunnions are vertical, the range setting becomes deflection setting and vice versa, and with the line of sight on the target the gun is actually laid at zero angle of elevation and at 5° in deflection. In any case, trunnion tilt operates to introduce into the deflection a component of the angle of elevation for which the sights are set, and to introduce into the angle of elevation a component of the deflection for which the sights are set. The component of deflection which is introduced into the angle of elevation by trunnion tilt is small enough to be neglected in any practical case. The component of elevation which affects the deflection may, however, cause very serious errors in deflection. The component of elevation which is lost due to trunnion tilt may, under exceptional circumstances, cause noticeable errors in range, but ordinarily these errors are small.

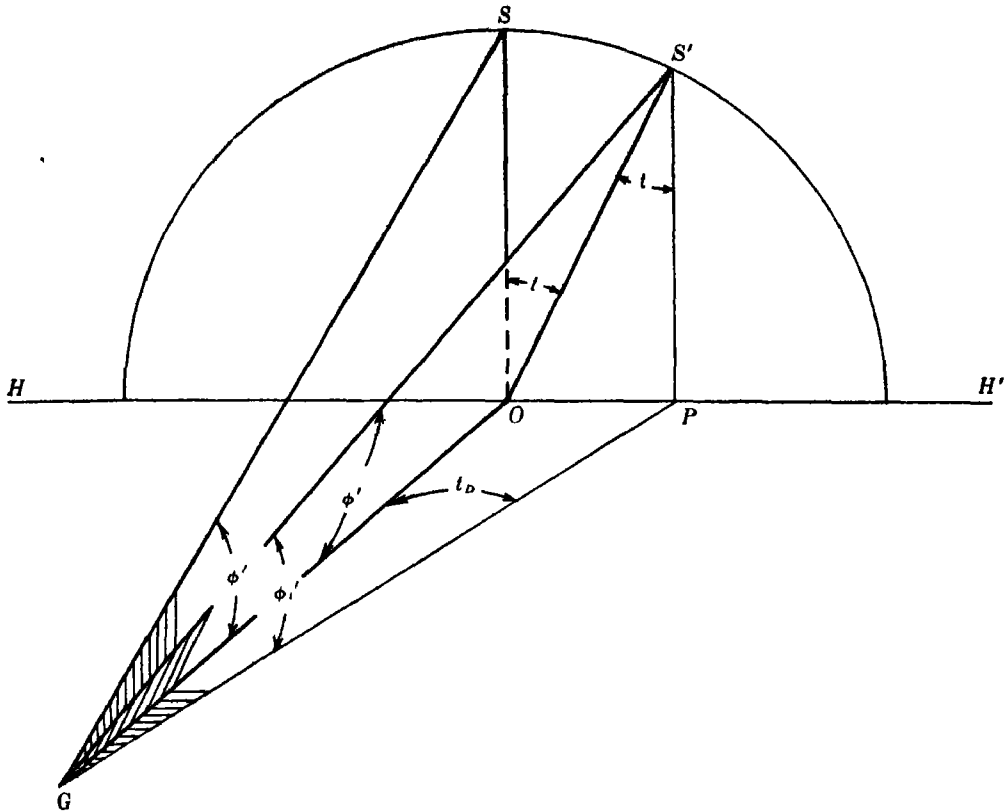


FIGURE 40

1312. In Figure 40, G represents the position of a gun and GO the line of sight; GS represents the axis of the bore with the trunnions of the gun in the horizontal plane HH' . The angle of elevation is OGS , or ϕ' . GS' represents the axis of the bore with the trunnions tilted to the right by the amount of the angle SOS' , or t . (These relations are shown in perspective.) The figure shows that, with the trunnions tilted, the angle of elevation is reduced to $S'GP$, or ϕ'_i , and a deflection component equal to the angle OGP , or t_D , is introduced. Noting that the triangles SOG , $S'OG$, $S'PG$, and POG all are right triangles, that $GS = GS'$ and $OS = OS'$, and that the angles OGS and OGS' both represent the angle of elevation ϕ' , we find the following relations.*

$$\sin \phi'_i = \frac{S'P}{GS'} = \frac{OS' \cos t}{OS' \csc \phi'}$$

whence

$$\sin \phi'_i = \sin \phi' \cos t. \quad (1308)$$

* These formulas, of course, do not take into account the angular deflection setting of the sights. Trunnion tilt operates to alter the deflection setting just as it operates to alter the elevation setting; that is, it reduces the effective deflection setting and introduces a component thereof into the angle of elevation. Within practical limits of deflection setting and trunnion tilt, however, the reduction in effective deflection setting is practically insignificant, and the component of deflection setting that is introduced into the angle of elevation is not likely to cause noticeable errors. By far the most important effect of trunnion tilt is the introduction of a component of the angle of elevation into the deflection, i.e., the effect represented by formula (1309) or (1310).

Also

Formulas for
effects of
trunnion tilt

$$\tan t_D = \frac{OP}{OG} = \frac{OS' \sin t}{OS' \cot \phi'}$$

whence

$$\tan t_D = \tan \phi' \sin t. \quad (1309)$$

From formula (1309) we can determine the angular deflection due to trunnion tilt; the corresponding linear deflection, which we shall denote by D_t , then is

$$D_t = X \tan t_D$$

or

$$D_t = X \tan \phi' \sin t. \quad (1310)$$

1313. The magnitude of the errors that result from trunnion tilt can be determined readily from formulas (1308) and (1310). Let us consider an extreme case, say a heavy cruiser firing practically deadahead, with gun selevated at 30° for a range of 28,000 yards, and rolling 10° to either side. If the guns are fired at the extremity of a roll, i.e., with $t = 10^\circ$, the angle of elevation is reduced by trunnion tilt to $29^\circ 30'$;

$$\sin \phi_t' = \sin 30^\circ \times \cos 10^\circ, \text{ or } \phi_t' = 29^\circ 30'$$

and the deflection error due to trunnion tilt is

$$D_t = 28,000 \times \tan 30^\circ \times \sin 10^\circ = \underline{2807 \text{ yards}}$$

The reduction in the angle of elevation is $30'$, which, for the battery of a heavy cruiser, corresponds to a reduction in range of about 200 yards at 28,000 yards. Hence it appears that the effect of trunnion tilt on the range is relatively small, even in this extreme case. The significant feature is that trunnion tilt, in this case, can cause deflection errors of any amount up to about 2800 yards (or about 100 miles) either to the right or to the left, depending upon the stage of the roll at the instant of firing.

The following example shows that trunnion tilt can cause very serious errors in deflection even under ordinary circumstances. Let us suppose that a $16''2600$ f.s. battery is being fired at a range of 25,000 yards, for which the angle of elevation is practically 17° , and under conditions involving angles of trunnion tilt as great as 2° to either side. From (1308) and (1310) we have

$$\sin \phi_t' = \sin 17^\circ \times \cos 2^\circ, \text{ or } \phi_t' = 16^\circ 59' 4''$$

$$D_t = 25,000 \times \tan 17^\circ \times \sin 2^\circ = \underline{267 \text{ yards}}$$

The maximum reduction in the angle of elevation, due to trunnion tilt, is therefore $0' 6''$, which corresponds to about 9 yards of range. The deflection error, however, can be as great as 267 yards (or about 11 miles) either to the right or to the left.

1314. Mechanical compensation for errors due to trunnion tilt is provided for in all director systems of recent design. However, there are still in service many director systems of earlier design which do not include this feature. In the absence of mechanical compensation for trunnion tilt errors, firing should be done only when the trunnions are practically horizontal. This applies, of course, to center fire, as well as to director fire with systems which are not equipped to

compensate mechanically for trunnion tilt. Although such procedure may entail a material sacrifice in the rate of fire, it should be obvious from the study made above that the alternative procedure of firing without regard for the existing condition of trunnion tilt may result in an extremely erratic deflection performance.*

COLD-GUN ERRORS

1315. It is well known, from practical experience, that a shot from a cold gun usually falls short with respect to succeeding shots which are fired from the same gun under otherwise identical conditions. The abnormal errors which occur on initial shots are naturally associated with the fact that the gun is cold for such shots, and they are accordingly called *cold-gun errors*. It is not a foregone conclusion, however, that the heating of the gun by the initial shot is the only effect which operates to change the range of succeeding shots, for if the heating of the gun were alone responsible for such changes one might reasonably expect the latter to occur with fair regularity at the firings of similar guns, or certainly at different firings of the same gun. That is to say, one might reasonably expect the cold-gun errors of all guns of the same caliber and initial velocity to be approximately the same, and the cold-gun errors of a given gun to remain approximately the same throughout the life of that gun. On the contrary, the performances of different guns of the same caliber and initial velocity, or of the same gun at different stages of its life, are actually found to differ materially in this respect. Although most guns settle down to a steady performance after the first shot, some do not settle down until several shots have been fired. In the case of the 4"2900 f.s. gun, for example, increases of range from shot to shot have been found to persist for as many as six shots. Also, there is strong evidence that some guns actually suffer *reductions* of range after the first shot.

Causes of cold-gun errors **1316.** It is probable that the change in the gun's performance after an initial round results from a combination of some or all of the following causes.

- (a) A portion of the energy of the charge is expended in heating the gun. This loss of energy is greatest for the first round, and probably diminishes rapidly thereafter. This cause, by itself, serves to explain the increases of range that often occur after the first round, but not the great variation in the amount of such increases for different guns of the same caliber and initial velocity, or for firings of the same gun at different stages of its life.
- (b) For the initial round the walls of the gun are at a fairly uniform temperature throughout their thickness; or the outside of the gun, if exposed to the rays of the sun, may be warmer than the bore. After the initial round, the bore is considerably warmer than the outside of the gun. It is not unlikely, therefore, that the structural parts of the gun readjust themselves after the bore becomes heated, and thus cause a change in the

* For any values of angle of elevation and trunnion tilt that are likely to occur in practice, the deflection error in mils due to trunnion tilt is approximately equal to one-third of the product of the angle of elevation and angle of trunnion tilt, each of the latter being expressed in degrees; that is, t_D (in mils) = $\frac{1}{3}\phi' \times t$ (in degrees). Applying this rule to the examples given in article 1313, we find for the first example a deflection error of about $\frac{1}{3} \times 30 \times 10 = 100$ mils, and for the second example a deflection error of about $\frac{1}{3} \times 17 \times 2 = 11$ mils.

droop. This cause can operate either to increase or to decrease the range, according to the manner in which the droop is affected.

- (c) The bore of a gun probably remains fairly well lubricated for the initial round, even though it has been washed out carefully prior to firing. The initial round not only removes all traces of lubrication from the bore but also fouls the latter with copper. The bore therefore offers less than the normal resistance to the passage of the projectile on the initial round, and the pressure, muzzle velocity, and range for this round are consequently less than normal (the effect being akin to that of erosion).
- (d) The temperature of the powder charge may be increased by exposure to the heat of the chamber, thus causing an increase of range after the initial round. Experiments have indicated, however, that the temperature of the powder charge is increased relatively little by exposure to the heat of the chamber for such periods of exposure as are likely to occur in practice.

1317. Fairly consistent shifts in deflection after the initial round, have been observed in the cases of some guns, and it is probable that such shifts occur for all guns, though they may often be small enough to escape notice. It would naturally be assumed that *increases* in range after the initial round would be accompanied by *increases* in drift, but observed results indicate that the drift may actually decrease after the initial round even though the range increases appreciably. In the case of the 5"3150 f.s. gun, for example, an *increase* in range and a shift to the *left* in deflection usually occur after the initial round, and there is evidence to indicate that the amount of the shift to the left in deflection varies approximately in proportion to the amount of the increase in range. Shifts to the left in deflection with increases in range, after the initial round, have also been noted in the case of the 16"2600 f.s. gun. It is to be presumed that a shift to the left after the initial round indicates that the latter suffered excessive drift. The combination of decreased range and increased drift which the initial round suffers in these cases can be accounted for satisfactorily only by irregular projection, as a result of which the drift is increased by a greater amplitude of precession and the range is decreased by a greater average obliquity of the projectile.

1318. Cold-gun errors are usually noticeable only at comparatively long ranges. When reliable information as to the amounts of these errors is available (and this can be gained only from careful analysis of previous firings), *cold-gun corrections* may be included in the ballistic corrections for the first salvo and removed after this salvo has been fired. In some cases the spotter is charged with the responsibility of making an allowance in his spot on the first salvo for the shift that is expected to occur after this salvo. However, the better practice is to handle the cold-gun correction altogether at the range-keeping station and to let the spotter spot for full due on the first salvo.

Typical examples of cold-gun errors

The cold-gun corrections for the 16"2600 f.s. gun, at battle ranges, have been found to vary from about *up* 100 yards in range and *left* 1 mil in deflection when the gun is new, to a maximum of about *up* 700 yards in range and *left* 4 mils in deflection when the gun has fired about 100 rounds. In the case of the 5"3150 f.s. gun at battle ranges, the cold-gun corrections have been found to vary from about *up* 150 yards in range and *left* 1 mil in deflection when the gun is new, to a maximum of about *up* 600 yards in range and *left* 4 mils in deflection when the gun has fired about 250

CHAPTER 14

THE ACCIDENTAL ERRORS OF GUNFIRE. THE APPLICATION OF THE LAWS OF PROBABILITY TO THESE ERRORS. DETERMINATION OF THE PROBABILITY OF HITTING A GIVEN TARGET.

New Symbols Introduced

- D True mean dispersion (article 1402 (f)); for range the subscript r is used and for deflection the subscript d .
- D' Apparent mean dispersion (article 1402 (e)); subscripts may be used as with D .
- a Distance, or error, of any given point from a reference point, such as the mean point of impact or the center of the hitting space (article 1413); subscripts are used as with D .
- d Error of the M.P.I.; subscripts are used as with D .
- S' Hitting space of a target (article 1402 (h)).
- w Width of a target across the line of fire.
- P Probability of hitting a given target; subscripts are used as with D . (The symbol P , however, may also be used to denote probability in connection with any other problem.)

1401. It is common knowledge that even under the best of conditions there is a certain amount of scattering, or dispersion, among the points of impact of shots from a gun. This dispersion obviously may be due to a multitude of causes. Even when a gun is stationary and rigidly fixed in elevation, variations in range and deflection are caused by differences among individual powder charges and projectiles, by variations from shot to shot in the conditions attending the projection and consequently affecting the flight of the projectile, and by variations from shot to shot in the atmosphere. In actual practice, additional variations are caused by errors in sight-setting and in aiming, and by variations from shot to shot in the motion of the gun due to rolling, pitching, and yawing of the ship. We shall, in this chapter, examine the nature and principal causes of dispersion, and determine how dispersion affects the control and effectiveness of gunfire.

Definitions 1402. The following definitions relate to terms which are used in connection with the analysis of fall of shot; these definitions are standard in the U. S. naval service and are used in connection with target-practice reports.

- | | |
|--|---|
| Salvo | (a) A <i>salvo</i> consists of two or more shots fired at the same target either simultaneously or within a short interval of time, such as the duration of a firing signal. |
| Mean point of impact, or M.P.I. | (b) The <i>mean point of impact</i> (abbreviated <i>M.P.I.</i>) of a salvo is the point which is at the geometrical center of the points of impact of the several shots of the salvo, <i>excluding wild shots</i> ; it is analogous to such terms as "center of figure," "center of location," etc., when referred to a group of points. |
| Pattern | (c) The <i>pattern in range</i> of a salvo is the distance, measured parallel to the line of fire, between the point of impact of the shot falling farthest beyond the M.P.I. and the point of impact of the shot |

falling farthest short of the M.P.I., *excluding wild shots*. The *pattern in deflection* of a salvo is the distance, measured at right angles to the line of fire, between the point of impact of the shot falling farthest to the right of the M.P.I. and the point of impact of the shot falling farthest to the left of the M.P.I., *excluding wild shots*. The term *pattern* has nothing to do with the arrangement of the several points of impact of a salvo, nor with the area covered by them. There is generally more occasion to deal with the pattern in range than with pattern in deflection, and hence the unqualified term *pattern* is often used when referring to the pattern in range.

Dispersion

- (d) The dispersion* of a shot is the distance between its point of impact and the M.P.I. of the salvo. The *dispersion in range* is the component of dispersion measured parallel to the line of fire, and the *dispersion in deflection* the component measured at right angles to the line of fire. The dispersion in range of a shot is called positive when the point of impact of the shot is beyond the M.P.I., and negative when the point of impact of the shot is short of the M.P.I. The dispersion in deflection of a shot is called positive when the point of impact of the shot is to the right of the M.P.I., and negative when the point of impact of the shot is to the left of the M.P.I. It is to be noted that the arithmetical sum of the greatest positive and negative dispersions in range is equal to the pattern in range, and that the arithmetical sum of the greatest positive and negative dispersions in deflection is equal to the pattern in deflection; also that the algebraic sum of the dispersions in range of the several shots of a salvo must equal zero, and likewise for the dispersions in deflection.

Apparent mean dispersion

- (e) The *apparent mean dispersion in range* (or *deflection*) of a salvo is the arithmetical mean of the dispersions in range (or deflection) of the several shots of that salvo, *excluding wild shots*.

True mean dispersion

- (f) The apparent mean dispersion is not a true measure of the accuracy of fire, since it is based on a limited number of observations. The true measure of accuracy is the mean dispersion of an infinitely great number of shots, all assumed to have been fired under as nearly as possible similar conditions, and it is called the *true mean dispersion*. Although it is impossible to measure the value of the true mean dispersion experimentally, a theoretical value is given by the relation

$$D = D' \sqrt{\frac{n}{n-1}} \quad (1401)^\dagger$$

* The term *dispersion*, as used here, is similar in meaning to the terms *error* and *deviation* as used by other services in the same connection. This usage of the term *dispersion* is standard in the U. S. naval service.

† It is shown in the laws of errors that if a mean error is reckoned from a finite number of observations, the true value (γ) of the mean error bears to the value (γ') of the mean error as reckoned from n observations, the relation

$$\gamma = \gamma' \sqrt{\frac{n}{n-1}}.$$

Proof of this proposition can be found in most treatises on the *Method of Least Squares*.

in which D is the true mean dispersion, D' the apparent mean dispersion, and n the number of shots from which D' has been measured. The following table gives values of the ratio of D to D' ,

i.e., values of the term $\sqrt{\frac{n}{n-1}}$, for salvos of from two to twelve

shots. The true mean dispersion is therefore defined as follows.

Ratios for
converting
apparent
mean dis-
persion to
true mean
dispersion

No. of shots in salvo	Ratio of true mean disper- sion to apparent mean dispersion
12	1.044
11	1.049
10	1.054
9	1.061
8	1.069
7	1.080
6	1.095
5	1.118
4	1.155
3	1.225
2	1.414

The *true mean dispersion in range (or deflection)* of a salvo is equal to the apparent mean dispersion in range (or deflection) of that salvo multiplied by a factor which depends on the number of shots in that salvo (excluding wild shots) and which is given in the above table.

Wild
shot

- (g) A *wild shot* is a shot whose dispersion is abnormal in comparison with the mean dispersion of the salvo in which that shot was fired. A shot may be *wild* either in range or in deflection.

The determination of how great a shot's individual dispersion may be in comparison with the mean dispersion of the salvo, before the shot may be classed as *wild*, is one of the most difficult problems in salvo analysis. Since wild shots are rejected in determining M.P.I.s, mean dispersion, and pattern sizes, it is evident that great care and good judgment must be exercised in determining whether a shot is truly *wild*, or whether the probable frequency of occurrence of a dispersion of the magnitude in question warrants the inclusion of the shot in the analysis. Otherwise the values derived may lead to very erroneous conclusions as to the performance analyzed.

There is considerable divergence among rules stated by various authorities for the rejection of wild shots. The general considerations upon which such rules are based will be treated later (art. 1503). But it may be noted here that in the U. S. naval service the rules to be used for determining wild shots for the purpose of target-practice analysis, are given in *Orders for Gunnery Exercises* and in *Gunnery Instructions*. They will not be stated here as they may vary from time to time.

or in the following works: art. 20, *Combination of Observations*, Brunt; art. 104, *The Calculus of Observations*, Whitaker and Robinson; art. 61, *Handbook of Ballistics*, Crans and Becker.

Hitting
space

- (h) The *hitting space* (S') is the distance, measured parallel to the line of fire, between the point of fall at the surface of the water of a shot that just pierces the top of the target and of one that just pierces the bottom of the target (i.e., the waterline on the engaged side of the target). The hitting space includes the projection of the target's vertical height upon the plane of the water and the target's horizontal dimension in the line of fire (we shall hereafter refer to the latter as the target's *depth* in the line of fire). Sometimes there is also included in the hitting space a distance in front of the target within which impacts are likely to produce either underwater hits or ricochet hits on the target. It is convenient to think of the hitting space as the pattern in range of the shortest and longest shots that can hit the target.

That portion of the hitting space which is due to the height of the target is approximately equal to $h \cot \omega$, where h is the height of the target and ω is the angle of fall. It can be found with sufficient approximation for practical purposes from either Column 7 or Column 19 of the range table, except for very short ranges (see arts. 916 and 1222).*

Error of
M.P.I.

- (i) The *error of the M.P.I.* is the distance of the M.P.I. from the center of the hitting space, measured parallel to the line of fire for range and at right angles to the line of fire for deflection

It is to be noted that the *error of the M.P.I.* of a salvo defines the error in location of the entire salvo with respect to the target, while the *mean dispersion* of the salvo relates to the errors of the several shots of the salvo with respect to their own M.P.I. and is entirely independent of the location of the salvo with respect to the target.

1403. Figure 41 is a plan view of a target TT resting on the surface of the water. The line of fire is normal to TT . The area outlined with shading represents the area behind the target within which shots piercing the target must fall. The dimension of this area in the line of fire is equal to the target's hitting space, and the dimension perpendicular to the line of fire is equal to the target's width; the center of the area is at C . The points of impact of a 10-shot salvo are represented by dots. The figure clearly illustrates how the pattern in range and pattern in deflection are measured. Assuming that the figure is plotted to a scale of 10 yards by 10 yards for each small square, we find that the pattern in range is 320 yards and the pattern in deflection 130 yards. (The target dimensions and values of pattern size and dispersion assumed for this illustration are not typical but have been selected for convenience in illustrating.)

The location of the M.P.I. is determined as follows. We shall refer each shot to the center of the target's waterline, denoted by P in the figure, and tabulate the amount it is over (+) or short (-), and right (+) or left (-), with respect to this point. Although any other reference point can be used for this purpose, the center of the target's waterline is usually the most convenient reference point.

We thus find that the M.P.I. is located 80 yards *over* and 40 yards *right* with respect to the center of the target's waterline, and it has been so plotted in Figure 41. The work is shown beneath Figure 41.

* The determination of the hitting space for very short ranges is treated as a special problem in Appendix B.

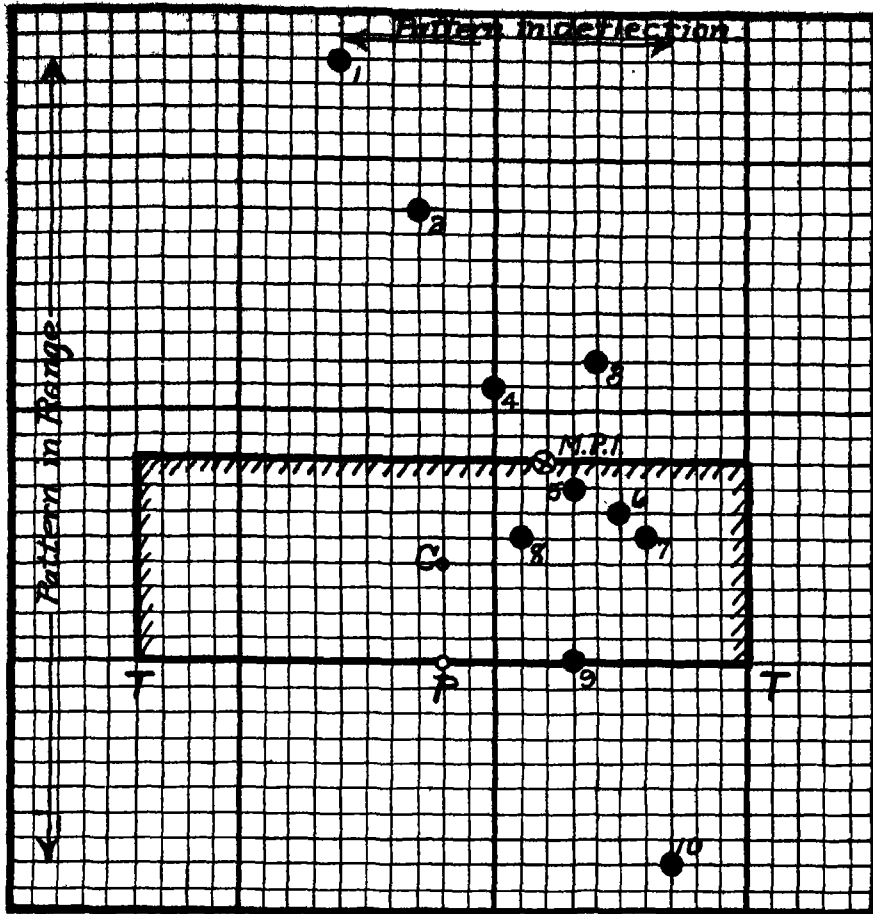


FIGURE 41

Determina-
tion of
M.P.I.,
pattern,
and
error of
M.P.I.

Shot No.	Distance from reference point			
	Range		Deflection	
	(+)	(-)	(+)	(-)
1	240			40
2	180			10
3	120		60	
4	110		20	
5	70		50	
6	60		70	
7	50		80	
8	50		30	
9		0	50	
10		80	90	
Sum.....	880	80	450	50
	80		50	
Difference...	800		400	
Mean.....	80		40	

The center of the hitting space, *C*, is located 40 yards *over* with respect to the center of the target's waterline, and the M.P.I. is therefore located 40 yards *over* and 40 yards *right* with respect to the center of the hitting space. According to definition, then, the *error of the M.P.I.* is 40 yards *over* and 40 yards *right*.

In determining the apparent mean dispersion it must be borne in mind that the apparent mean dispersion is the *arithmetical* mean of the dispersions of the several shots. To find the apparent mean dispersion in range we therefore take the sum of the distances the several shots are *over* or *short* with respect to the M.P.I. and divide it by the number of shots in the salvo; to find the apparent mean dispersion in deflection we proceed similarly, using the distances the several shots are to the right or left of the M.P.I. We thus find that the apparent mean dispersion is 66 yards in range and 32 yards in deflection.

Determination
of apparent
and true
mean dis-
persion

Shot No	Dispersion	
	In Range	In Deflection
1	160	80
2	100	50
3	40	20
4	30	20
5	10	10
6	20	30
7	30	40
8	30	10
9	80	10
10	160	50
Sum	660	320
Mean	66 yds.	32 yds.

The true mean dispersion is now found by multiplying the apparent mean dispersion by 1.054, which is the conversion factor for a salvo of 10 shots (as given in the table in article 1402 (f)). Hence the true mean dispersion is $66 \times 1.054 = 70$ yards in range, and $32 \times 1.054 = 34$ yards in deflection.

An erroneous process, often used to shorten the determination of apparent mean dispersion, is to *decrease* the sum of the several distances "over" the reference point by the product of the number of "overs" and the distance the M. P. I. itself is "over" this same point, and to *increase* the sum of the several distances "short" by the product of the number of "shorts" and the distance the M. P. I. is "over" the reference point, and then to add these two quantities and divide by the number of shots, etc. Referring to the example illustrated above, it would appear that this last-described process would be the equivalent of individually decreasing each "over" and increasing each "short" and adding all the distances so found and dividing by the number of shots, which is, indeed, what has been done in finding the apparent mean dispersion above. But applying this supposedly equivalent process to the above example, we get for the apparent mean dispersion in range only 48 yards and in deflection only 26 yards, amounts so far in error as to be utterly worthless. The reason for the discrepancy lies in the fact that when handling the sums in this manner we are not getting the *arithmetical* sum in all cases. For instance, it will be noted that the dispersions in range of shots 5, 6, 7, and 8 of the example are 10, 20, 30, and 30 yards, respectively. If we reduce the sum of the eight "overs" of the example by 8×80 , we are, in fact, treating shots 5, 6, 7, and 8 as if their

dispersions were $(-)10$, $(-)20$, $(-)30$, and $(-)30$, respectively. But the dispersion of shot 5 is $70 \sim 80 = 10$, and not $(-)10$, and similarly for shots 6, 7, and 8 the dispersions should, as always, have the plus sign, while by using the sum of all the errors we actually would be using these dispersions with the minus sign. The shortened process will fail to give the correct result whenever the salvo contains impacts which are located between the M. P. I. and the reference point from which the individual shots are measured, and as this is almost always the case, the correct method illustrated in this article should be employed habitually.

1404. The M.P.I. may also be referred to the vertical plane. The M.P.I. of hits in vertical portions of a target (as for example in a target-screen) can be found readily by the process that has already been illustrated for finding the M.P.I. in the horizontal plane. But if some shots fail to hit in vertical portions of a target, it is then necessary to project their impacts from the horizontal plane to the vertical plane, which can be done by means of Column 19 of the range table. In any event, the M.P.I. in the horizontal plane having been found, it is a simple matter to project it to the vertical plane by means of Column 19. It will be observed that in the example in Figure 41 the M.P.I. in the vertical plane evidently is at the top edge of the target.

By projecting the several impacts of a salvo to the vertical plane we may, by the process illustrated in article 1403, determine the mean dispersion in the vertical plane. However, the mean dispersion in range in the vertical plane, which is called the *mean vertical dispersion*, can also be found readily from the mean dispersion in range in the horizontal plane, by means of Column 19. The mean dispersion in deflection in the vertical plane, which is called the *mean lateral dispersion*, is practically the same as the mean dispersion in deflection in the horizontal plane.

1405. Even at the proving ground, where special facilities are available for ensuring uniformity of powder charges and projectiles and of their loading, for laying the gun accurately in elevation, and for securing uniformity in many other factors upon which the trajectory depends, unaccountable variations in range and deflection occur. In order to understand this it must be accepted, first of all, that it is never possible to measure with absolute certainty, and hence never possible to control completely, all of the factors which influence the flight of a projectile. For example, although great care is exercised to ensure uniformity as to the weight and temperature of the powder charge, the initial velocity nevertheless may vary by small amounts from shot to shot due to unmeasurable and uncontrollable variations in the powder charge. Similarly, although great care may be exercised in laying a gun at exactly the same elevation for a series of shots, the angle of departure nevertheless may vary from shot to shot due to variations in droop and jump. It is likely that irregularities in projection are among the principal sources of unaccountable errors. Such irregularities may result not only from differences in the manner in which the bore of the gun reacts to the pressures, temperatures and shock of firing, but also from differences in the smoothness with which the projectile itself passes through the bore and hence in the position of its axis at the instant of projection, and from differences in the action of the muzzle blast on the projectile. Differences in the action of the rotating band as the projectile passes through the bore, and in the condition of the rotating band during the flight of the projectile, may have a very marked effect on the dispersion. If the band does not engrave properly the pro-

jectile receives insufficient rotation and its flight may be erratic. Erratic flight may result from deformation of the rotating band (such as fringing), or from shearing of the band.

1406. In the service firing of guns dispersion results also from other causes in addition to those which have been outlined above. Differences occur in bore-sighting and in the lining up of director systems, due both to personal errors and to limitations of material. Powder charges may suffer alteration to varying degrees due to differences in the conditions of storage in different magazines. The temperatures of individual powder charges may differ among one another due to differences in the temperatures of the various magazines in which the charges are stored; sometimes charges are stored in gun compartments or in ready-service lockers on deck for fairly long periods prior to firing, and very material differences in powder temperature may be occasioned thereby. Variations in shell seating cause variations in the density of loading and have a marked effect on dispersion. Then there are differences among the several guns of a battery,—differences in erosion for which only approximate compensation can be made, differences in droop, and differences in mounting (for example, differences in trunnion tilt among the several guns of a battery due to differences among the inclinations of their respective roller-paths).

Further causes of dispersion are occasioned by the motions of the firing ship and target. Unsteadiness of the gun platform causes dispersion among the shots of a salvo due not only to variable pointing errors, but also to variations in the delay between the closing of the firing key and firing of the gun (such variations may be occasioned by differences in the firing circuits, primers, or ignition pads, or by differences in the location of the powder bag with respect to the primer vent, or even by differences in the condition of the primer vent). Relative motion between the ship and target gives rise to variable control errors, such as errors in range-finding, errors in estimation of target course and speed, errors in sight-setting, etc.

1407. The errors which have been discussed above, as well as many others which contribute to dispersion in gunfire, come under the general classification of *accidental errors*, a classification which applies not merely to errors of gunfire but to occurrences in general. **Accidental errors** are just as likely to be positive as negative, and they are more likely to be small than large. They follow the laws of probability and their behavior, in the long run can be predicted under the terms of these laws. Knowledge as to the behavior of the accidental errors of gunfire is of importance not only for estimating the effectiveness of gunfire under various conditions, but also for establishing sound doctrines for spotting.

The laws of probability deal with the prediction of future occurrences on the basis of information gained from past occurrences. Such occurrences may be of various kinds. Thus the laws of probability provide a basis for predictions as to the probable remaining duration of life of a person of a given age, the predictions being based on statistics taken in the past as to the frequency of occurrence of deaths at various ages beyond the given age. Such predictions are the basis of mortality tables and hence the foundation of all life-insurance schemes. Probability laws have an important part in the science of statistics and may, in this connection, apply to almost any kind of event, or occurrence. In gunnery, the occurrences to which the laws of probability are applied are the accidental errors

of gunfire, and the predictions deal with the probable sizes and distribution of such errors.

Probability laws vary considerably according to the nature of the occurrences to which they pertain.* The law which pertains to symmetrically distributed accidental errors is known as the *normal probability law*. The accidental errors of gunfire generally follow a fairly symmetrical distribution, and the normal probability law is applied to them.

1408. The normal probability law is based upon the following considerations:

1. That positive and negative errors of the same size are equally probable and hence will occur with equal frequencies.
2. That small errors are more probable than large errors and hence will occur with greater frequency than larger errors.
3. That very large errors will not occur (or, more specifically, that very large errors will probably be due to mistakes and not be classifiable as accidental errors).

These considerations all pertain to the probable frequency of occurrence of errors of certain size and sign. If we translate them into a mathematical formula, or equation, the curve representing that equation will have the following characteristics (abscissas measuring the size of error and ordinates the frequency of their occurrence):

1. Since positive and negative errors of the same size occur with equal frequencies, the curve must be symmetrical to the right and left of the point that denotes zero error (i.e., $x=0$), which is the origin.
2. Since small errors are more frequent than large errors, the maximum ordinate must occur at the point that denotes the least error (i.e., $x=0$), which is the origin.
3. Since large errors occur but rarely, and beyond some finite but undefinable limit do not occur at all, the right- and left-hand branches of the curve must each approach the horizontal axis rapidly and meet it at some undefinable point. These features are illustrated in the curves shown in Figure 42.

The equation representing the probability law must take into account the actual frequency with which each size of error will occur. The derivation of this equation (i.e., for the *normal probability law*) may be found in any treatise on the *Method of Least Squares*;† it will be given here without further proof.

Normal
Probability
Equation

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2} \quad (1402)$$

In equation (1402), x and y represent abscissas and ordinates of the curve, the former being a measure of the size of errors and the latter of the frequency of their occurrence; e is the base of the Naperian system of logarithms (approximately 2.7183), and h is a constant determined from the particular set of observations to which a particular curve is to apply.

1409. The constant h is a measure of the precision of the instrument with which the observed results are obtained; we may call it the *index of precision*. As applied to gunfire, h is the index of precision of

* A thorough and interesting treatment of a wide variety of applications of the laws of probability is given in *Probability and its Engineering Uses*, by Thornton C. Fry.

† Ref. Chapter IV, *The Theory of Errors and Method of Least Squares*, W. W. Johnson; also, Chapter II, *The Combination of Observations*, Brunt. A very complete treatment, with particular reference to applications in gunnery, may be found in §58-60, *Handbook of Ballistics*, Cranz and Becker.

the gun. An index of precision may be defined in several ways, all of which, however, depend upon analysis of results actually given by the instrument to which the index applies.

In the *Theory of Least Squares* it is demonstrated that the most accurate *mean error* of a series of observed quantities from their *arithmetical mean*, is that which results from taking the square root of the sum of the squares of the several observed errors. This mean is usually called the *quadratic mean error*, and it is almost invariably used in connection with problems dealing with precision of measurements. In these cases the quadratic mean error is the index of precision.

In applications to the errors of gunfire it is found more convenient and sufficiently accurate to employ the arithmetical mean, or average, obtained by the simple process of adding together the various errors, without regard to sign, and dividing the sum by the number of errors.* The latter quantity is properly called the *arithmetical mean error* but, since it is so commonly used in gunnery, it is usually referred to simply as the *mean error* or the *mean dispersion*. It is shown in the derivation of the normal probability equation, that if the measure of precision is taken as the arithmetical mean error of the gun, or D , then the index of precision h is related to D by the equation,†

$$h = \frac{1}{D\sqrt{\pi}}. \quad (1403)$$

Substituting this value of h in (1402) we have the probability equation in the form in which it is commonly used in gunnery,

$$y = \frac{1}{\pi D} e^{-x^2/\pi D^2} \quad (1404)$$

and in this equation D represents the *true mean dispersion*, as defined in article 1402 (f), and the other symbols have meanings as already noted under equation (1402).

1410. The curve representing the normal probability equation is known as the *normal probability curve*. Its general form is illustrated in Figure 42, and it will be noted that the general characteristics of the curve conform to the considerations upon which the normal probability law is based (article 1408), with the exception that the branches of the curve do not actually meet the horizontal axis. For finite considerations, however, this condition is met, for the curve, while not actually meeting the X -axis except at infinity, approaches this axis very closely within relatively small finite distances from the origin.

While every normal probability curve must have the general characteristics noted in article 1408 and illustrated in Figure 42, the actual form of a particular curve depends upon the value of the index of precision h used in its equation, or, when dealing with our own problem, the mean dispersion D establishes the actual form of the particular curve. This will be understood when we consider that the index of precision is a measure of the mean error of the observations, and that if the mean error is small the several errors must be closely grouped about their mean; thus with a small

* Ref. §63, *Handbook of Ballistics*, Cranz and Becker, for statements and examples as to the relative accuracy of these "means" Ref. also articles 17-21, *The Combination of Observations*, Brunt.

† Ref. §60, p. 374, *Handbook of Ballistics*, Cranz and Becker, and article 17, *The Combination of Observations*, Brunt.

mean error the frequency of small errors is increased, the maximum ordinate of the curve becomes great, and its branches descend steeply to the horizontal axis. With a large mean error the several errors are more widely dispersed, the central portion of the curve is lower and flatter, and the branches approach the horizontal axis less steeply.

Curve A of Figure 42 illustrates a normal probability curve based upon an index of relatively *great precision*, while curve B is based on an index of relatively *low precision*. Or the former curve might be said to represent the normal probability curve for a gun of relatively *small mean dispersion* and the other to represent relatively *great mean dispersion*. Actually curve A represents a mean dispersion just

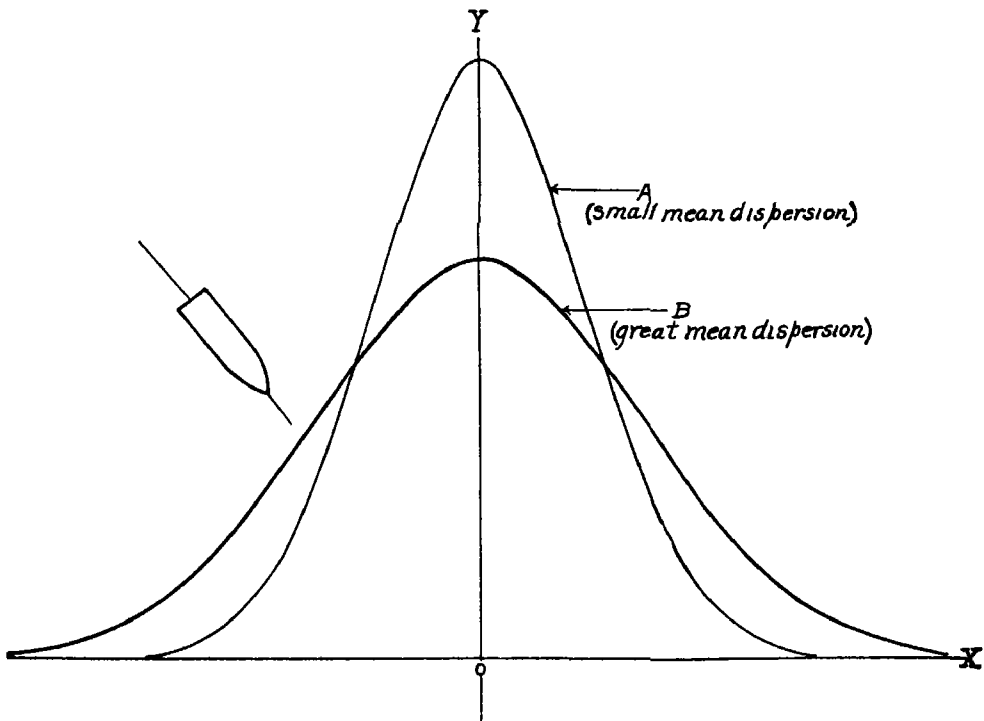


FIGURE 42

two-thirds as great as represented in curve B, both curves being drawn to the same scale.

The general nature of the probability curve may be understood readily if we consider the following illustration. Let us suppose that *O*, in Figure 42, is a point for which we have accurately determined the mean hitting range, and that we fire a great many shots at this point; also let us suppose, for the purpose of this illustration, that each projectile remains exactly where it first strikes. The projectiles would then pile up around the point *O*, which would be their M. P. I. under the conditions we have assumed. If the gun used were a very accurate gun, having a very small mean dispersion, the projectiles would evidently be heaped high around the M. P. I. If the gun were one with large mean dispersion, the pile would be lower and flatter, and the projectiles would straggle out to greater distances from the center of the pile. The contour of a cross section of the pile taken parallel to the line of fire would evidently be the probability curve for that particular gun in range; and the cross section taken normal to the line of fire would represent the probability

curve in deflection. Thus it is seen that the probability curve really gives us a very easily understandable picture of the shot distribution of a gun.

1411. Let us suppose, now, that we desire to estimate our chances of hitting within a given distance of O , in Figure 42, with the same gun with which the experiment described in the previous article was assumed to have been made. Let us deal first with distance in range and assume 10 yards for the amount. If we were to count the number of projectiles that had fallen at ranges not greater than 10 yards beyond O , nor less than 10 yards short of O , and to find the ratio of this number to the total number of projectiles, we would have a numerical measure of our chances of hitting within ± 10 yards of O in range. If 1000 shots had been fired and 600 were counted in the given strip, we would have 60% of shots with ranges within ± 10 yards of O . We speak of this percentage as the *probability* of obtaining that result, and the probability of having a range error within ± 10 yards in this case would be .60. Now if we continued to fire at point O with the same gun, the probability of hitting the given strip being .60, of the next 10 shots 6 would *probably* hit the strip, and so would 60 of the next 100 *probably* hit it, etc. We cannot say that such results would *certainly* be obtained, but on the basis of the best information we have, those results would be the most probable ones and the ones that, in the long run, would probably be fulfilled.

Assuming a similar strip centered at O , but bounded by lines 10 yards to the right and 10 yards to the left of O , and applying the same process as above, let us suppose that a count showed 900 shots to lie within this strip. Then the probability of hitting within 10 yards *right or left* of O would be .90, and what has been said of the ratio in connection with the range applies equally well here.

We have assumed above that 60% of the projectiles fell at ranges within ± 10 yards of O , and 90% within distances 10 yards right or left of O . Then it follows that of the 60% lying within ± 10 yards of O in range, only 90% will also lie within ± 10 yards of O in deflection, whence $90\% \times 60\%$, or 54%, will lie within the square measuring 20 yards on each side and centered at O . Thus we find that the probability of hitting that square is .54.

1412. The probability equation gives us a formula by means of which we may arrive at probability ratios such as illustrated in the above example. For the example of the previous article it would tell us what percentage of shots would probably fall within any given limits with respect to the M.P.I., or O . In order to give this result for range it must employ as the value of D the true mean dispersion in range, and for deflection the true mean dispersion in deflection.

Since the probability equation expresses the frequency y with which an error of size x will occur, to find the frequency with which all errors from 0 to $\pm a$ will occur we must integrate the equation within the limits $x = +a$ and $x = -a$. Thus the frequency of occurrence of errors within $\pm a$, when the mean dispersion is D , becomes

$$\frac{1}{\pi D} \int_{-a}^{+a} e^{-x^2/2D^2} dx. \quad (1405)$$

Equation (1404) has been so constructed that its integral between the limits $-\infty$ and $+\infty$ equals unity, and hence (1405) expresses a fraction, or percentage, of unity or, as illustrated in the example above, a probability.* Thus (1405) gives the

* If the integral of the curve between any chosen limits is to express the probability of occurrence of errors within those limits, then by extending the limits to ∞ in both directions we will include all errors, and the probability of any errors falling within these infinite limits becomes *certainly*, or unity, whence the integral extended to these limits must equal unity.

probability of the occurrence of errors within $+a$ and $-a$ when the true mean dispersion is D .

Referring to Figure 43, this means that the total area under the curve of (1404) is unity, while the area of the strip extending from $+a$ to $-a$ is given by (1405); and that, since the total area equals unity, the area of the strip shown will be found directly as a percentage of the whole. Thus by integrating the probability equation we have merely done by a formal mathematical process what we have done in the example of article 1411; i.e., we have made a *count* of the number of errors within $\pm a$ and found the ratio of this number to the whole.

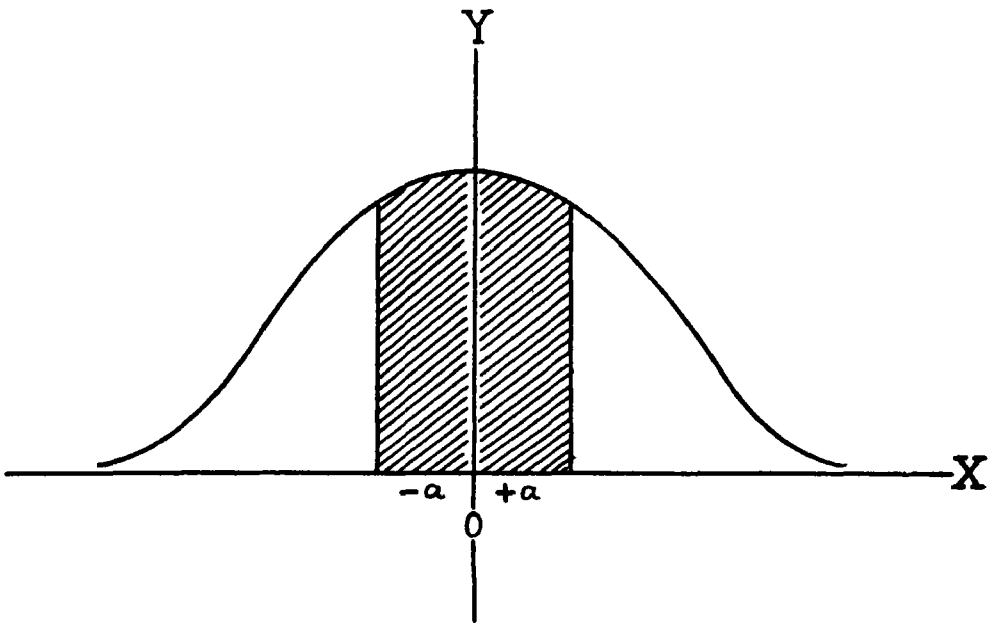


FIGURE 43

1413. In order to eliminate the necessity of integrating (1405) for each individual problem, it is convenient to tabulate the values of the integral for various *ratios* of x to D . If we were to tabulate the values of the integral for various values of x by itself, we would evidently have to provide a different table for each value of D . But if we evaluate the integral for various *ratios* of x to D , we need make but a single table, the entering argument for which will then be the *ratio* x/D . Thus we can tabulate the value of the integral for the value of x which is equal to $1/2D$, or $x/D = 1/2$; this value of the integral will apply equally well to a case in which the particular error (x) dealt with is ± 100 yards and the true mean dispersion (D) 200 yards, and to a case in which the particular error is ± 10 yards and the mean dispersion 20 yards, and to any case in which the particular error is half as great as the mean dispersion. And the same applies to any other ratio of x to D for which we evaluate the integral.

When dealing with particular values of the errors, we represent them by the symbol a , and the ratios referred to above then are expressed as a/D . The table of values of the normal probability integral in terms of the ratio a/D is given below; we shall refer to it as the *Probability Table*. The entering argument for the probability table is the ratio a/D , or the ratio of the particular error ($\pm a$) with which we are dealing, to the index of precision of the instrument with which we are dealing, which, in the case of gunfire, is the true mean dispersion (D). The quantity (P) found from the table is the probability that an error not greater than $\pm a$ will occur.

PROBABILITY OF AN ERROR NOT GREATER THAN $\pm a$
IN TERMS OF THE RATIO $\frac{a}{D}$

Probability Table	$\frac{a}{D}$	P	$\frac{a}{D}$	P	$\frac{a}{D}$	P	$\frac{a}{D}$	P
	0.1	.064	1.1	.620	2.1	.906	3.1	.987
	0.2	.127	1.2	.662	2.2	.921	3.2	.990
	0.3	.189	1.3	.700	2.3	.934	3.3	.992
	0.4	.250	1.4	.735	2.4	.945	3.4	.994
	0.5	.310	1.5	.768	2.5	.954	3.5	.995
	0.6	.368	1.6	.798	2.6	.962	3.6	.996
	0.7	.424	1.7	.825	2.7	.969	3.7	.997
	0.8	.477	1.8	.849	2.8	.974	3.8	.998
	0.9	.527	1.9	.870	2.9	.979	3.9	.998
	1.0	.575	2.0	.889	3.0	.983	4.0	.999

An examination of the probability table will show that for the ratio $a/D = 4$, the probability is $P = .999$; this means that there is but one chance in a thousand that any future result will be in error by an amount that is greater than about four times the arithmetical mean error of past results. Applying this to our own problem it means that the dispersion of any individual shot from a gun has but one chance in a thousand of being greater than four times the mean dispersion of the gun. Evidently this feature has an important bearing on the determination of wild shots. If we assume that any shot which does not fall within the limits which should contain 99% of all the shots of a series or salvo, is a wild shot, then, according to the table, any shot whose dispersion is greater than 3.2 times the true mean dispersion is a wild shot (for when $P = .99$, $a/D = 3.2$). If we extend the limits to include 99.9% of the shots, a dispersion greater than 4 times the true mean dispersion defines a wild shot. The difficulty lies in setting a logical limit when dealing with small numbers of shots, and this matter will be discussed further later on (article 1503). But the point to be noted here is that whatever limits are chosen, the ratio of greatest individual dispersion to true mean dispersion that will conform to these limits may be determined readily from the probability table.

1414. If we enter the probability table with the value $P = .500$ we will find that the corresponding value of a/D is .846. This means that if $\pm a = .846D$, the probability of occurrence of an error within $\pm a$ is just $1/2$; i.e., it is just as likely that an error *smaller* than $\pm a$ will occur as that an error *larger* than $\pm a$ will occur. For this reason the error of size $\pm .846D$ is called the *probable error* in many services and this term is found in many textbooks and reference books. The term "probable error" is arbitrarily used to denote what might be called the "50-50 error."

If an error of $\pm a = .846D$ will not be exceeded just half the time, then the total width of the band that will be hit just half the time is $2 \times .846D$, or $1.692D$. This

The 50% zone band is called the *50% zone*. In terms of hitting space and mean dispersion, this merely means that in order to receive 50% of hits the hitting space of the target must equal 1.692 times the mean dispersion of the gun. In some services gun errors are commonly spoken of in terms of their 50% zones, and in most others in terms of the "probable error," tables being provided which may be entered directly with the "probable error." In our service the term "probable error" is not used, but the term "50% zone" is used in target practice reports. The latter may be found from the relation noted above, viz.,

$$50\% \text{ zone} = 1.692D. \quad (1406)$$

1415. An important consideration in any use of the laws of probability is that the value of any prediction based upon them depends largely upon the number of occurrences to which the prediction is applied. For example, let us assume a mean dispersion of 50 yards and find what the probability is that with this mean dispersion shots will fall within ± 100 yards of the M. P. I. The ratio a/D equals $100/50$, or 2.0, in this case, whence we find from the table that $P = .889$. Thus the answer is that the probability of obtaining this result is practically .90, or that about 90% of the shots should fall within ± 100 yards of the M. P. I. If we apply this percentage to 3-gun salvos, we get 2.7 hits per salvo within the chosen limits as the probable number of hits per salvo. We can never have 2.7 hits on a salvo, hence with 3-gun salvos we will not get that predicted percentage of hits on any one salvo. If we get 2 hits, the percentage for that salvo will be only 67%, varying very considerably from 90%. Obviously even approximate agreement with the predicted percentage will depend upon firing a considerable number of salvos, and ultimately our total percentage of hits will be about 90%. With 10-gun salvos we could reasonably expect the majority of the several salvos to produce 9 hits each (within the given limits referred to the M. P. I.).

Although these features are obvious enough when we pause to consider the fundamental nature of the laws of probability, they are often lost sight of in practice, and failure to give them due weight often leads to the suspicion that the laws of probability are worthless theories. The value of these laws lies in their application "in the long run." Application of doctrines based on them will produce net profits just as surely as the zero and double zero on a roulette wheel will produce a profit for the bank, "in the long run."

1416. We may define probability as the numerical measure of the expectation that a certain occurrence will take place. We have developed methods for determining this numerical measure for various sets of assumed conditions. Thus, in the preceding article, we found that for the assumed set of conditions $a = \pm 100$ and $D = 50$, the corresponding value of P is .889.

It is an axiom of the laws of probability that if an event is independently controlled by more than one set of conditions, the probability of its occurrence under all of these sets of conditions is equal to the product of the several probabilities of its occurrence under the several sets of conditions. This may be illustrated in many ways. For instance, if there is one chance in two that a man A will be at the spot X at the instant Y , and one chance in two that a man B will be at the spot X at the instant Y , then the chance that A will meet B at the spot X at the instant Y is evidently one in four. We can express this also as follows: the probability that A will be at X at the instant Y is $P_1 = \frac{1}{2}$, and that B will be there at the same instant is $P_2 = \frac{1}{2}$, hence the probability that both will be there at the same instant is $P_1 \times P_2 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. Again, we know that the probability of drawing the king of

**Product of
separate
probabilities**

spades from a full deck of cards, in one trial, is $P = \frac{1}{52}$, because there is but one king of spades and there are fifty-two cards. The same result may be obtained in various other ways. The probability of drawing *any* card of the spade suit is $P_1 = \frac{1}{4}$, since one-fourth of all of the cards are of the spade suit. The probability that a card of the spade suit, if drawn, will be the *king*, is $P_2 = \frac{1}{13}$, since there is only one king among the thirteen cards of the spade suit. Hence the probability that the card drawn will be both a spade and a king is $P = P_1 \times P_2 = \frac{1}{4} \times \frac{1}{13} = \frac{1}{52}$. Or we have the probability of drawing *any* spade, $P_1 = \frac{1}{4}$, of drawing any *face* card of the spade suit, $P_2 = \frac{3}{13}$, and the probability that any face card will be the *king*, $P_3 = \frac{1}{3}$; whence the probability of drawing the king of spades is $P = P_1 \times P_2 \times P_3 = \frac{1}{4} \times \frac{3}{13} \times \frac{1}{3} = \frac{1}{52}$. The example of article 1411 illustrated how this axiom applies to our problem. It simply means that if the probability of hitting in range is P_r and the probability of hitting in deflection is P_d , then the final probability of hitting is $P_r \times P_d$.

Another axiom of these laws is that if P is the probability that an event *will* occur, then $1 - P$ is the probability that it *will not* occur. This is easily understood when we think of probability as a fraction of unity, or a percentage. Thus if we find that the probable percentage of hits is $100P$, the probable percentage of misses evidently must be $100 - 100P$, or $100(1 - P)$.

1417. We are now prepared to deal with some practical applications of the theories we have discussed. Let us investigate the applications to the determination of the probability of hitting a target whose dimensions are known, with a gun whose true mean dispersion is known. Under the assumption that the accidental errors of gunfire follow the distribution indicated by the normal probability equation (and illustrated by the normal probability curve), it is evident that the probability of hitting a given target is greatest when the M.P.I. is located at the center of the target's hitting space. Under this condition the densest portion of the salvo, being symmetrically disposed about the M.P.I., will be centered on the hitting space of the target.* Thus when we speak of the maximum percentage of hits that will probably be made by a certain gun against a certain target, we ordinarily mean the probable percentage of hits that will occur when the M.P.I. is located at the center of the hitting space. The latter condition is, however, by no means one that we may assume as typical in actual service.

If the M.P.I. is located at the center of the hitting space, the allowable error in range (over or short) within which hits will occur, will evidently be one-half the hitting space. We will let S' denote the hitting space (the prime being used to distinguish it from the symbol for danger space). Then under the conditions assumed to govern the maximum percentage of hits, we have $\pm a = \frac{S'}{2}$, and with any given value D for the true mean dispersion, the probability of hitting will be that value of P which will be obtained by entering the probability table with $\frac{a}{D}$, and the percentage of hits will be $100P$. A few problems will illustrate these features.†

* See article 1425.

† For the sake of simplicity the hitting areas are treated, in these problems, as rectangles. Actually they are irregular, although somewhat resembling rectangles with rounded corners. The deck area itself is approximately elliptical, and equal roughly to 80% of the area of the enclosing rectangle (i.e., length times beam). However, the total hitting area, which includes the areas short of and beyond the actual limits of the ship as well as the deck area itself, equals roughly 90% of the enclosing rectangle (i.e., total hitting space times width across line of fire) at long ranges, and a greater proportion at short ranges. It may be con-

1418. In the problems given here, as well as elsewhere in this chapter, the values of mean dispersion and of pattern size are purely fictitious, this being necessary because real values of such quantities are information of confidential nature. The problems should therefore be viewed merely as illustrations of methods, and not as being designed to illustrate typical results.

1. *Given:* Eight-gun salvos are fired from a ship with its main battery of 16"2600 f.s. guns. The range to the target is 20,000 yards; the freeboard of the target is 28 feet, its beam 90 feet, its length 600 feet, and it is lying perpendicular to the line of fire. It is assumed that shots striking not more than 15 yards in front of the target will make effective under-water hits. The true mean dispersion of the guns is 200 yards in range and 30 yards in deflection.

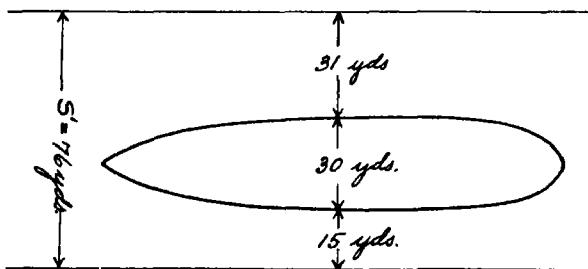
Find: The probable percentage of hits that will be made under the conditions given, and with the M.P.I. at the center of the hitting space.

In accordance with article 1402 (h), the portion of the hitting space due to the height (or freeboard) of the target may be found from Column 7 of the range table, and is,

$$\frac{28}{20} \times 22 = 30.8, \text{ or } 31 \text{ yards.}^*$$

To this must be added the depth of the target in the line of fire (in this case its beam) and the distance in front of the target within which hits will be effective. This gives as the total hitting space,

$$S' = 31 + \frac{90}{3} + 15 = \underline{76 \text{ yards}}$$



With the M.P.I. centered in this hitting space, we will have as the maximum allowable variation in range, $\pm \frac{76}{2} = \pm 38$ yards, whence for range $a_r = \pm 38$ yards; likewise the maximum variation to right and left of the center will be *one-half* the length of target, or $\pm \frac{300}{3}$ yards, whence for deflection $a_d = \pm 100$ yards.

sidered, therefore, that the probabilities obtained in these problems are of the order of about 10% too great. Considering the general uncertainty of data that enter into problems of this character, it is doubtful whether anything is to be gained by striving for a more exact treatment of the target dimensions. However, methods for dealing with areas of various shapes are outlined in §§ 68-69, *Handbook of Ballistics*, Vol. I, Cranz and Becker.

* It is sufficiently accurate for any practical purpose to take such computations as these to the nearest whole yard or, in the case of the ratio $\frac{a}{D}$, to two decimal places

For the sake of clarity, we may give subscripts to the symbol for true mean dispersion (as has been done with a above), and thus for range $D_r=200$ and for deflection $D_d=30$. Applying this subscript notation also to the symbol P , we have,

For range, $\frac{a_r}{D_r} = \frac{38}{200} = .19$, whence $P_r = .121$.

For deflection, $\frac{a_d}{D_d} = \frac{100}{30} = 3.33$, whence $P_d = .993$.

Then, in accordance with article 1416, the probability of hitting the target is,

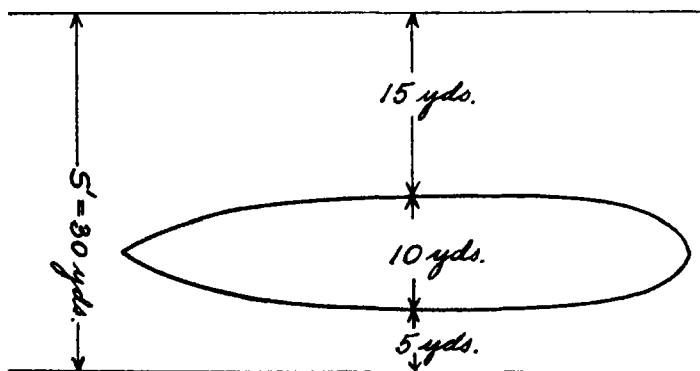
$$P = P_r \times P_d = .121 \times .993 = .120$$

or the probable percentage of hits is 12.0%, giving about one hit per salvo.

2. *Given:* Ten-gun salvos are fired from a ship with its battery of 5"3150 f.s. guns. The range to the target is 12,000 yards; the freeboard of the target is 16 feet, its beam 30 feet, its length 300 feet, and it is lying perpendicular to the line of fire. It is assumed that shots striking not more than 5 yards in front of the target will make effective under-water hits. The true mean dispersion of the guns is 150 yards in range and 25 yards in deflection.

Find: The probable percentage of hits that will be made under the conditions given, and with the M.P.I. at the center of the hitting space.

Proceeding exactly as in the first problem, we have,



$$S' = \left(\frac{16}{20} \times 19 \right) + \frac{30}{3} + 5 = \underline{30 \text{ yards}}$$

whence,

$$a_r = \pm \frac{30}{2} = \pm 15 \text{ yards.}$$

Also,

$$a_d = \pm \frac{1}{2} \left(\frac{300}{3} \right) = \pm 50 \text{ yards.}$$

$D_r = 150$ yards (as given)

$D_d = 25$ yards (as given)

For range, $\frac{a_r}{D_r} = \frac{15}{150} = .10$, whence $P_r = .064$.

For deflection, $\frac{a_d}{D_d} = \frac{50}{25} = 2.00$, whence $P_d = .889$.

And

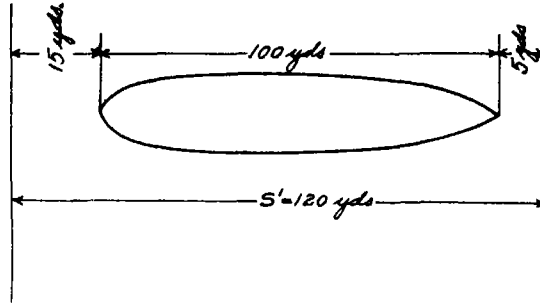
$$P = P_r \times P_d = .064 \times .889 = .057$$

or the probable percentage of hits is 5.7%, giving about one hit in two salvos.

3. *Given:* The same data as for problem 2 above, except that the target is steaming directly toward the firing ship.

Find: The probable percentage of hits, with the M.P.I. at the center of the hitting space.

Noting that the depth of the target *in* the line of fire is now measured by the target's length, while the target's beam becomes the width *across* the line of fire, we may proceed just as before.



$$S' = \left(\frac{16}{20} \times 19 \right) + \frac{300}{3} + 5 = \underline{120 \text{ yards}}$$

whence,

$$a_r = \pm \frac{120}{2} = \pm 60 \text{ yards}$$

Also,

$$a_d = \pm \frac{1}{2} \left(\frac{30}{3} \right) = \pm 5 \text{ yards}$$

$$D_r = 150 \text{ yards (as given)}$$

$$D_d = 25 \text{ yards (as given)}$$

For range, $\frac{a_r}{D_r} = \frac{60}{150} = .40$, whence $P_r = .250$.

For deflection, $\frac{a_d}{D_d} = \frac{5}{25} = .20$, whence $P_d = .127$.

And

$$P = P_r \times P_d = .250 \times .127 = .032$$

or the probable percentage of hits is 3.2%, giving about one hit in three salvos.

1419. The examples dealt with in the preceding article were all based on the assumption that the M.P.I. coincides with the center of the hitting space. This condition will be met only when control errors have been reduced to zero, which can be considered no more likely than that the gun errors (or dispersion) can be reduced to zero. It is necessary, therefore, to consider methods for determining the probability of hitting when the M.P.I. is *not* at the center of the hitting space; i.e., with an error of M.P.I., and this error may, of course, be an error either in range or in deflection or in both. This problem involves no new principles but merely a further manipulation of the same process already applied to the simpler case.

Probability of hitting, when M.P.I. is not at center of hitting space

It will be noted that the probability table gives us directly the *probability of an error not greater than $\pm a$ in terms of the ratio $\frac{a}{D}$* (article 1413). This means that we get directly the probability that an error will be included within limits which are *symmetrical* with respect to the point from which they are measured. In the examples of the preceding article this direct solution was applicable because the limits within which hits were to be made were also symmetrical with respect to the M.P.I. If, however, we move the M.P.I. from the center of the hitting space, this ceases to be true.

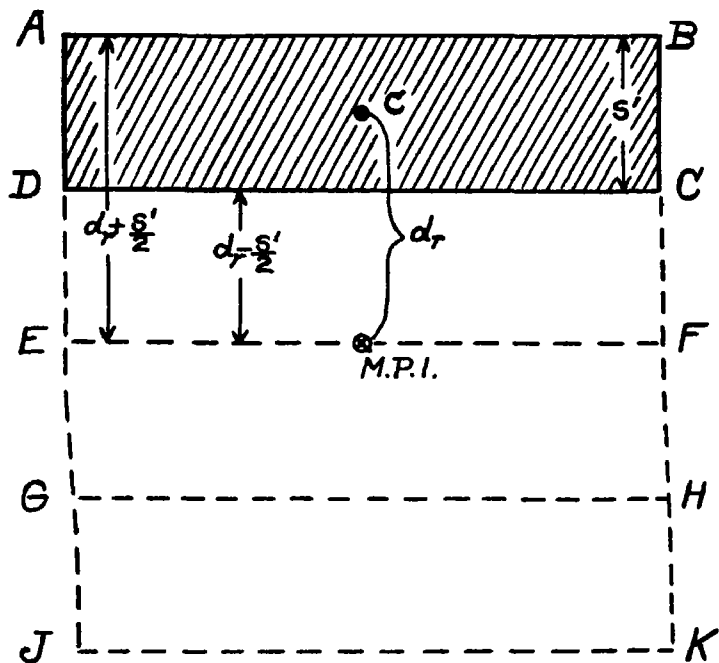


FIGURE 44

Referring to Figure 44, let the rectangle $ABCD$ denote the hitting space, with center at C . Now if the M.P.I. is also at C , then all shots falling within $\pm \frac{S'}{2}$ of C will be hits in range. In the above examples this condition gave us the value $a = \frac{S'}{2}$.

With the same hitting space $ABCD$ (Figure 44), let us now move the M.P.I. to a position away from C . In the figure it is shown at a distance d_r , directly in front of the center of the hitting space (C). Thus the error of the M.P.I. is d_r , in range, and for the present we will consider it to have no error in deflection. The limits within which shots will be hits in range are now no longer equal distances on either side of the M.P.I. Reference to the figure will show that the limits are now the distances $(d_r + \frac{S'}{2})$ and $(d_r - \frac{S'}{2})$, both beyond the M.P.I. In other words, there will fall within the hitting space only those shots whose several dispersions beyond the M.P.I. are not more than $(d_r + \frac{S'}{2})$ nor less than $(d_r - \frac{S'}{2})$; no shots falling short of the M.P.I. can hit the area $ABCD$. But if we use the limit $(d_r + \frac{S'}{2})$ for $\pm a$, and with the ratio $\frac{a}{D}$ so obtained enter the probability table, we will find the probability of hitting within distances lying *either* short of or beyond

the M.P.I. the amount of this a ; or, as the figure shows, we will find the probability of hitting within the area $ABKJ$ which is symmetrical about the M.P.I. In like manner, if we use the limit $(d_r - \frac{S'}{2})$ for $\pm a$, we will find the probability of hitting within the area $DCHG$ which is symmetrical about the M.P.I. If we subtract the area $DCHG$ from the area $ABKJ$, we will have left the two areas $ABCD$ and $GHKJ$, and from the construction of the figure it is evident that the latter two areas are equal. Thus it follows that if we find the probabilities of hitting the larger area $ABKJ$ and the smaller area $DCHG$, one-half the difference between these probabilities will represent the probability of hitting the area $ABCD$.

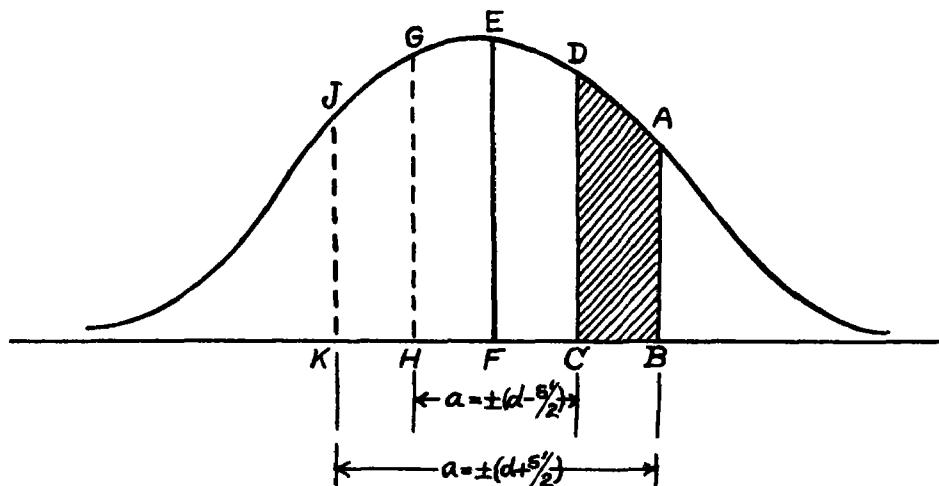


FIGURE 45

In Figure 45 the same features are illustrated in terms of areas under the probability curve. Since the probability table actually consists of a tabulation of areas under this curve between various limits, Figure 45 closely illustrates the actual process employed in this problem.

1420. An illustration entirely similar to the above might be made for the case when the M.P.I. has an error d_d in deflection, and no error in range; it would simply be a repetition of what has been said for the case with an error of M.P.I. only in range, with the substitution of the appropriate symbols for the deflection error. If we turn Figure 44 left through 90° , so that the M.P.I. lies to the right of the shaded area $ABCD$, and if we substitute for S' the symbol w to denote *width* of the target *across* the line of fire, and for d_r the symbol d_d to denote the error of the M.P.I. in deflection, the figure will illustrate the problem of finding the probability of hitting with an error of M.P.I. in deflection.

And, finally, if we consider the M.P.I. to have errors both in range and in deflection, the problem involves a combination of the two elementary cases. Figure 46 illustrates this problem. The M.P.I. is located with an error in range d_r , short, and an error in deflection d_d left, of the center C of the hitting area.

Carrying through the process with the range error, we will find the probability of hitting within the uppermost strip (of depth S'), which contains our hitting area. Carrying through the process with the deflection error, we will find the probability of hitting within the farthest right-hand strip (of width w), which also contains our hitting area. The probability of hitting the area itself is then determined by taking the product of the probabilities in range and deflection.

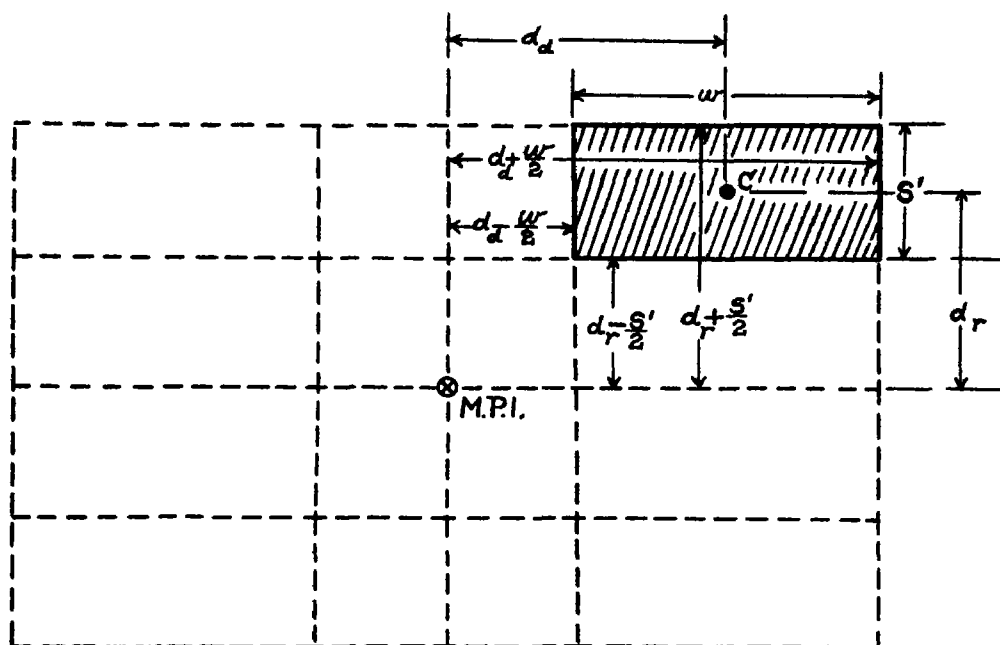


FIGURE 46

We may now state the following rules:

Process of determining probability of hitting, when the M.P.I. has an error both in range and in deflection

ting the target is:

If a target's hitting space is S' and its width across the line of fire is w , and if the mean dispersion of the gun is D_r in range and D_d in deflection, and if the error of the M.P.I. is d_r in range and d_d in deflection, then the probability of hitting the target is:

(a) In range.

$$P_r = \frac{1}{2}(P_{r_1} - P_{r_2})$$

in which P_{r_1} is found from the probability table with $\pm a_{r_1} = (d_r + \frac{S'}{2})$ and $D = D_r$, and P_{r_2} is found from the probability table with $\pm a_{r_2} = (d_r - \frac{S'}{2})^*$ and $D = D_r$.

(b) In deflection:

$$P_d = \frac{1}{2}(P_{d_1} - P_{d_2})$$

in which P_{d_1} is found from the probability table with $\pm a_{d_1} = (d_d + \frac{w}{2})$ and $D = D_d$, and P_{d_2} is found from the probability table with $\pm a_{d_2} = (d_d - \frac{w}{2})^*$ and $D = D_d$.

(c) In both range and deflection:

$$P = P_r \times P_d$$

* If $\frac{S'}{2} > d_r$ or $\frac{w}{2} > d_d$, which will cause a_{r_2} or a_{d_2} to be negative, respectively, the quantities P_{r_2} or P_{d_2} are additive to P_{r_1} or P_{d_1} , respectively. This case arises whenever the M.P.I. lies within the hitting area, and that we must take one-half the sum of the two P 's in this case may be verified by drawing the figure.

1421. The following problems illustrate the rules stated above.

1. *Given:* Ten-gun salvos are fired from a ship with its battery of 5"3150 f.s. guns. The range to the target is 12,000 yards; the freeboard of the target is 16 feet, its beam 30 feet, its length 300 feet, and it is lying perpendicular to the line of fire. It is assumed that shots striking not more than 5 yards in front of the target will make effective under-water hits. The true mean dispersion of the guns is 150 yards in range and 25 yards in deflection.

Find: The probable percentage of hits that will be made under the conditions given, and with an error of M.P.I. of 200 yards in range and 40 yards in deflection.

The hitting space is (see sketch for problem 2, article 1418),

$$S' = \left(\frac{16}{20} \times 19 \right) + \frac{30}{3} + 5 = \underline{30 \text{ yards}}$$

The width of the target across the line of fire is,

$$w = \frac{300}{3} = \underline{100 \text{ yards}}$$

The following values have been given,

$$\begin{array}{ll} d_r = 200 \text{ yards} & D_r = 150 \text{ yards} \\ d_a = 40 \text{ yards} & D_a = 25 \text{ yards.} \end{array}$$

Now, following the steps laid down in article 1420, we have:

$$\begin{aligned} \text{(a)} \quad a_{r_1} &= \left(d_r + \frac{S'}{2} \right) = 200 + 15 = \underline{215 \text{ yards}} \\ \frac{a_{r_1}}{D_r} &= \frac{215}{150} = 1.43, \text{ whence } P_{r_1} = \underline{.745} \\ a_{r_2} &= \left(d_r - \frac{S'}{2} \right) = 200 - 15 = \underline{185 \text{ yards}} \\ \frac{a_{r_2}}{D_r} &= \frac{185}{150} = 1.23, \text{ whence } P_{r_2} = \underline{.673} \\ P_r &= \frac{1}{2}(P_{r_1} - P_{r_2}) = \frac{1}{2}(.745 - .673) = \underline{.036} \\ \text{(b)} \quad a_{d_1} &= \left(d_a + \frac{w}{2} \right) = 40 + 50 = \underline{90 \text{ yards}} \\ \frac{a_{d_1}}{D_a} &= \frac{90}{25} = 3.60, \text{ whence } P_{d_1} = \underline{.996} \\ a_{d_2} &= \left(d_a - \frac{w}{2} \right) = 40 - 50 = \underline{(-)10 \text{ yards}} \\ \frac{a_{d_2}}{D_a} &= \frac{10}{25} = (-)0.40, \text{ whence } P_{d_2} = \underline{(-).250} \\ P_d &= \frac{1}{2}(P_{d_1} - P_{d_2}) = \frac{1}{2}(.996 + .250)^* = \underline{.623} \\ \text{(c)} \quad P &= P_r \times P_d = (.036 \times .623) = \underline{.022} \end{aligned}$$

Thus the probable percentage of hits is 2.2%, giving about one hit in five salvos.

* See note on page 224, which applies in this instance.

2. *Given:* The same data as for problem 1 above, except that the target is steaming directly toward the firing ship.

Find: The probable percentage of hits, the error of M.P.I. being also as in problem 1, above.

We will proceed at once to list our data. (See sketch for problem 3 article 1418.)

$$S' = \left(\frac{16}{20} \times 19 \right) + \frac{300}{3} + 5 = \underline{120 \text{ yards}}$$

$$w = \frac{30}{3} = \underline{10 \text{ yards}}$$

$$d_r = 200 \text{ yards} \qquad D_r = 150 \text{ yards}$$

$$d_d = 40 \text{ yards} \qquad D_d = 25 \text{ yards}$$

And the solution follows, according to article 1420.

$$(a) \qquad a_{r_1} = \left(d_r + \frac{S'}{2} \right) = 200 + 60 = \underline{260 \text{ yards}}$$

$$\frac{a_{r_1}}{D_r} = \frac{260}{150} = 1.73, \text{ whence } P_{r_1} = \underline{.832}$$

$$a_{r_2} = \left(d_r - \frac{S'}{2} \right) = 200 - 60 = \underline{140 \text{ yards}}$$

$$\frac{a_{r_2}}{D_r} = \frac{140}{150} = 0.93, \text{ whence } P_{r_2} = \underline{.541}$$

$$P_r = \frac{1}{2}(P_{r_1} - P_{r_2}) = \frac{1}{2}(.832 - .541) = \underline{.146}$$

$$(b) \qquad a_{d_1} = \left(d_d + \frac{w}{2} \right) = 40 + 5 = \underline{45 \text{ yards}}$$

$$\frac{a_{d_1}}{D_d} = \frac{45}{25} = 1.80, \text{ whence } P_{d_1} = \underline{.849}$$

$$a_{d_2} = \left(d_d - \frac{w}{2} \right) = 40 - 5 = \underline{35 \text{ yards}}$$

$$\frac{a_{d_2}}{D_d} = \frac{35}{25} = 1.40, \text{ whence } P_{d_2} = \underline{.735}$$

$$P_d = \frac{1}{2}(P_{d_1} - P_{d_2}) = \frac{1}{2}(.849 - .735) = \underline{.057}$$

$$(c) \qquad P = P_r \times P_d = (.146 \times .057) = \underline{.008}.$$

Thus the probable percentage of hits is 0.8%, giving about one hit in twelve salvos. (Compare this result with that of problem 1 above, and with those of problems 2 and 3 under article 1418.)

1422. Having determined the probable percentage of hits for any given set of conditions, it follows directly, from the axiom stated in article 1416, that the percentage of misses will equal one-hundred minus the percentage of hits. It may also be determined what proportions of these misses will be "shorts," "overs," "rights" or "lefts." The determination of the number of "shorts" that are likely

to occur is information useful to the spotter, for at long ranges the spotter on board ship sees little of what goes on beyond the target and must depend to a great extent upon what he sees in front of the target. "Shorts," when referred to in this connection, are considered to be impacts entirely short of the target, not merely short of the center of the hitting space.

Let us consider again problem 1 in article 1418. We found there that with the conditions as given, and with the M.P.I. at the center of the hitting space, about one hit per salvo would be probable. Now with the M.P.I. at the center of the hitting space, the misses would be equally distributed short of and beyond the target (i.e., its hitting space). Hence one-half of the misses would be "shorts," and one-half "overs." Under the conditions of the particular problem being considered, we would have $\frac{1}{2} \times (8-1)$ "shorts," or from 3 to 4 "shorts." Thus the spotter would know that salvos which showed from 3 to 4 "shorts" were properly centered over the target.

How many "shorts" will occur, theoretically, with various errors of M.P.I. in range, may also be determined readily enough. It will be of interest to consider how many hits will be probable when the hitting space lies just within the end of the pattern nearest the spotter. The latter condition will evidently result when the M.P.I. lies just one-half pattern beyond the engaged side of the target, or, assuming that the hitting space is bounded by the waterline on the engaged side of the target, when the error of the M.P.I. equals one-half the pattern minus one-half the hitting space. The following two examples will illustrate the method of handling this problem and also furnish the results for further discussion.

1. *Given:* A battery of 12 guns fires in salvo at a target whose total hitting space equals 80 yards (no allowance for distance in front of target for under-water hits is included in this figure). The true mean dispersion of the guns is 115 yards in range.

Find: How many "shorts" will probably occur when the M.P.I. coincides with the center of the hitting space?

$$S' = 80 \text{ yards (as given)}$$

$$\text{whence, } a_r = 40 \text{ yards.}$$

$$\text{Also, } D_r = 115 \text{ yards (as given)}$$

$$\text{whence, } \frac{a_r}{D_r} = \frac{40}{115} = .35, \text{ and } P_r = .219.$$

Therefore the probable percentage of hits is 21.9%, which gives about 2 or 3 hits per salvo and about 9 or 10 misses per salvo. Of the 9 or 10 misses per salvo one-half should be "shorts," hence the probable number of "shorts" when the M.P.I. coincides with the center of the hitting space, is about 5.

2. *Given:* The same data as for problem 1 above, and in addition that the average pattern size of the 12-gun salvos is 500 yards.

Find: The probable percentage of hits when the M.P.I. is located 250 yards beyond the engaged side of the target.

$$S' = 80 \text{ yards (as given)}$$

and the error of M.P.I., being measured from the center of the hitting space, is,

$$d_r = 250 - \frac{S'}{2} = \underline{210 \text{ yards}}$$

Also,

$$D_r = 115 \text{ yards (as given).}$$

Applying the rules of article 1420 (and noting that deflection does not enter into the problem), we have,

$$a_{r_1} = \left(d_r + \frac{S'}{2} \right) = 210 + 40 = \underline{250 \text{ yards}}$$

$$\frac{a_{r_1}}{D_r} = \frac{250}{115} = 2.17, \text{ whence } P_{r_1} = \underline{.917}$$

$$a_{r_2} = \left(d_r - \frac{S'}{2} \right) = 210 - 40 = \underline{170 \text{ yards}}$$

$$\frac{a_{r_2}}{D_r} = \frac{170}{115} = 1.48, \text{ whence } P_{r_2} = \underline{.761}$$

$$P_r = \frac{1}{2}(P_{r_1} + P_{r_2}) = \frac{1}{2}(.917 + .761) = \underline{.839}$$

Therefore, with the M.P.I. just one-half a pattern beyond the waterline on the engaged side of the target, the probable percentage of hits is 7.8%, giving about *one* hit per salvo.

1423. The results found in the second problem of the preceding article are of interest because they represent the best that a spotter can hope for when he sees no "shorts." Probably the most disastrous fault committed by spotters is that of feeling that "shorts" are wasted and should be spotted up to the target. Except at very close ranges, some "shorts" must necessarily occur (on the average) when the M.P.I. is at the center of the hitting space. The greater the range becomes, the more closely the theoretical percentage of "shorts" approaches 50%. Spotters should know, in general terms, how this percentage varies with range for the battery they spot.

Problems such as the above are not offered here as an attempt to establish an absolute spotting gage, but rather to provide a foundation for correct reasoning by spotters. The briefest of attention to the features illustrated above, and the roughest of computations, should make it apparent that the spotter who at battle ranges fails to keep short (on the average) a considerable portion of his shots, is sacrificing hits—certainly not gaining hits. And that the spotter who indulges the temptation to put the salvo all "just 'over' and in the hitting space," is an idle dreamer.

1424. It will be recalled that in article 1407 reference was made to the fact that the normal probability law is applicable to such errors as follow a symmetrical distribution, and it was also noted there that the accidental errors of gunfire are principally of this type. At short and moderate ranges these errors may be considered, for all practical purposes, to occur symmetrically. At long ranges (i.e., ranges resulting from high angles of elevation) the flight characteristics of the projectile operate to cause a measurable dissymmetry in the distribution of errors. The flight characteristics of a projectile are associated with its stability in flight, and are among the causes of dispersion. It appears that for the comparatively short duration of flight at short and moderate ranges, the errors due to variations in flight stability are relatively small and perhaps also practically symmetrical—or, being small themselves, any dissymmetry among them has no appreciable effect upon the resultant of many other elemental errors which are symmetrical. But at

Unsymmetrical errors at long ranges

long ranges the errors due to variations in flight stability become relatively greater. That they should then also be unsymmetrical in range seems plausible when one considers that large *reductions* in range may result from "wobbling," whereas it is less likely that similarly great *increases* due to abnormally good flight will occur. Thus among the many elemental errors contributing to the inherent errors of gunfire, it appears that those due to variations in the projectile's stability in flight are unsymmetrical, and at long ranges, when these errors may become of controlling magnitude, the resultant total errors may also become unsymmetrical to a measurable degree.

In salvo firing at long ranges, the features discussed above sometimes result in a marked straggling of shots at the "short" end of the pattern, and in bunching at the farther end. The M.P.I. of such a salvo would obviously be located *beyond* the center of the pattern, there would be more impacts *beyond* the M.P.I. than short of it, and the average dispersion of the impacts beyond the M.P.I. would be *less* than the average dispersion of those short of it. The normal law, and the normal probability table, would seem not to be applicable to such errors as these, for the normal law is based on symmetrical distribution, which means an equal number of impacts short of and beyond the M.P.I., and average dispersions of equal amount short of and beyond the M.P.I.

The fact that such visible discrepancies are often noted between conditions actually occurring and the theoretical conditions upon which the normal probability law is based, often leads to skepticism as to the worth of any conclusions that may be derived from the applications of this law. Unsymmetrical distributions are common in many other branches of research in which probability laws are used. In the science of statistics unsymmetrical distributions are the rule rather than the exception. A great many special probability laws have been devised to handle such cases, some of which differ very radically from the normal law.* If the deviation of the errors of gunfire from the assumptions governing the normal law were sufficiently great, a special law might be devised for them. But a closer investigation of the situation does *not* indicate that such discrepancies as exist warrant the discarding of the normal law.

1425. In order to illustrate further some of the features just discussed, as well as to provide examples of how theoretical results obtained by application of the normal probability law compare with practical results, two analyses based on recent firings will be presented.†

Illustration of an unsymmetrical range distribution In Figure 47, curve A represents the actual distribution in range of 200 shots fired at a proving ground with various 6" guns and various types of projectiles. The firings were made in series of from three to ten shots. For each of these series the M.P.I., and the dispersions of the several shots from the M.P.I., were determined. As the mean ranges of the various series differed considerably, the dispersions of the several shots were expressed as percentages of the ranges of their respective M.P.I.s, and these various dispersions were then plotted with reference to a common M.P.I., as shown in the figure. The firings used included a variety of types of

* Ref. Chapter VIII, *Probability and Its Engineering Uses*, by Thornton C. Fry. Also Chapter IX, *The Combination of Observations*, by Brunt. Also, articles 90 and 160, *The Calculus of Observations*, by Whitaker and Robinson.

† The actual values of the dispersions are not given, but the curves are drawn to scale and the value of D to the same scale is shown in order that the distribution may be studied in terms of errors compared to D .

projectile and of rifling, and were taken at random from data recorded in a service publication. Altogether the data may be regarded as representing conditions unfavorable, as compared to conditions usually encountered, for testing the applicability of the normal probability law. However, such data have been used here in order that some of the features previously discussed may appear in exaggerated degree for purposes of discussion.

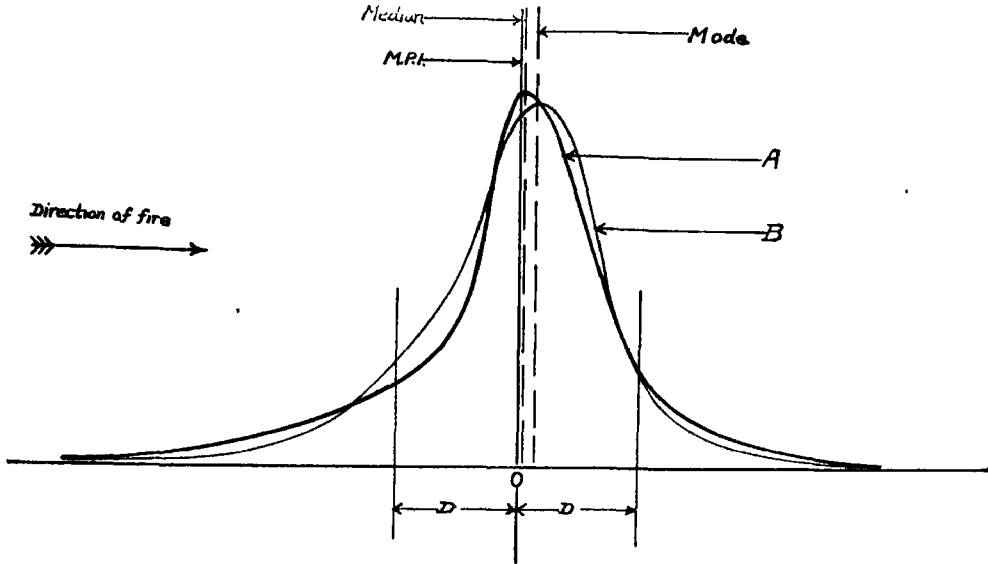


FIGURE 47

It will be noted that curve A, although having some of the general characteristics of the normal probability curve, is markedly lopsided. It falls off from its peak more steeply short of the M.P.I. than beyond it; but short of the M.P.I. it flattens out sooner and ultimately approaches the X-axis less rapidly than beyond the M.P.I. This indicates that shots were bunched more closely beyond the M.P.I. than short of it, and that there were more large errors short of the M.P.I. than beyond it. It indicates that the *density* of impacts decreased less rapidly beyond the M.P.I. than short of it; i.e., that the M.P.I. was not centered in the region where impacts were densest.

If the M.P.I. is not centered in the densest area of impacts, it is evident that the maximum percentage of hits will not occur with the M.P.I. at the center of the hitting space, although the latter assumption was made in arriving at the results in the problems of article 1418. Moreover, the normal probability table would not serve in this case, for it gives results only on the basis of a symmetrical error ($\pm a$).

To secure the greatest percentage of hits it is evidently necessary to place the salvo so that the point which represents the center of greatest density of impacts coincides with the center of the hitting space. With reference to frequency distributions in general, such a point is called the *mode*,* and the *mode* may be defined, for our case, as that point within the salvo pattern on either side of which the density of impacts

The "mode" in frequency distributions

* The terms *mode* and *median* are used principally in statistical work. Ref: article 61, *The Combination of Observations*, Brunt.

The "median" in frequency distributions systematically decreases. Another reference point in such distributions is known as the *median*, which is defined for our case as that point which has an equal number of impacts on either side of it. It is found that, on the average, the *median* of a frequency distribution lies between the M.P.I. and the *mode*.*

Returning now to Figure 47, we find illustrated the *mode* and the *median*; the *mode* was determined by finding the axis about which the peak of the distribution curve becomes symmetrical, and this new curve (*B*) has also been drawn. Curve *B* is seen to be much more nearly symmetrical at the peak than curve *A*, and it is, on the whole, a better approximation of the normal probability curve, although the left branch still shows markedly the straggling short. Thus it appears that the situation can be improved by placing the *mode* at the center of the hitting space and reckoning mean dispersion and probability of hitting from the *mode*. But when we consider the magnitude of the variation between these two axes of reference, it becomes apparent that such a refinement is unnecessary for any practical purpose. In Figure 47 the size of the true mean dispersion of the shots represented there, is indicated to scale. The distance between the M.P.I. and *mode* is about $\frac{1}{4}D$. The mean dispersion from the *mode* is about 2.5% less than the mean dispersion from the M.P.I.

For the conditions represented in the above data, the straightforward application of the normal probability table as outlined in the problems of article 1418, would then cause discrepancies due to an error of about 2.5% in the value of *D* used, and to an error of about $\frac{1}{4}D$ in the location of the M.P.I.—i.e., we would find a result based on a mean dispersion slightly too great and on an M.P.I. a short distance from the center of the hitting space instead of exactly at the center. And as this applies to a none too favorable case, it seems that the normal law works perhaps even better than we would naturally have expected it to work.

1426. It follows from the above discussion that in cases of unsymmetrical fall of shot the *mode*, rather than the M.P.I., should be spotted to the target.

Spotting the "mode" Spotting the *mode* is not only the better but also the simpler procedure, since it amounts simply to spotting the densest portion of the salvo to the target. It is often difficult, in unsymmetrical distributions, to locate the M.P.I. quickly by eye.

Another interesting feature brought out by this discussion is that the *median*, for unsymmetrical distributions, lies closer to the *mode* than does the M.P.I. Thus

Spotting the "median" a spotter who simply spots the 50-50 point of the salvo is, on the average, doing the logical thing—he is actually doing better than he would do if he tried to spot the M.P.I. For aircraft spotters, who can readily count the impacts of a salvo, the spotting of the *median* is easily the simplest method and at the same time probably the most certain. For symmetrical distributions the M.P.I., *median* and *mode* are, of course, identical, hence these observations apply equally well to either case.†

But the discussion in article 1415 again becomes pertinent here. The rules derived here may readily be discredited by applying them to individual instances; any rules that might be devised could likewise be discredited in such a manner. It should be borne in mind that rules based on the laws of probability are designed to serve best "in the long run"; they will prove best in *most* instances occurring, not necessarily in all.

* Ref: p. 121, *An Introduction to the Theory of Statistics*, Yule (1919).

† Ref. also article 102, *The Calculus of Observations*, by Whitaker and Robinson.

1427. The second analysis is based on the fall of shot that occurred in the long-range battle practices of a division of three ships; the range may be considered as representative of modern battle ranges. Altogether, 134 shots are represented in this analysis, including all shots fired by the three ships on their respective practices, with the exception, in each case, of the first salvo. The ships were equipped with identical batteries and their individual mean ranges were approximately equal, so that the fall of shot could readily be referred to a common M.P.I. by finding the M.P.I. of each salvo and the several dispersions of the shots from their respective M.P.I.s, as was done in the first illustration. The ships selected for this example were not selected because there was any reason to believe

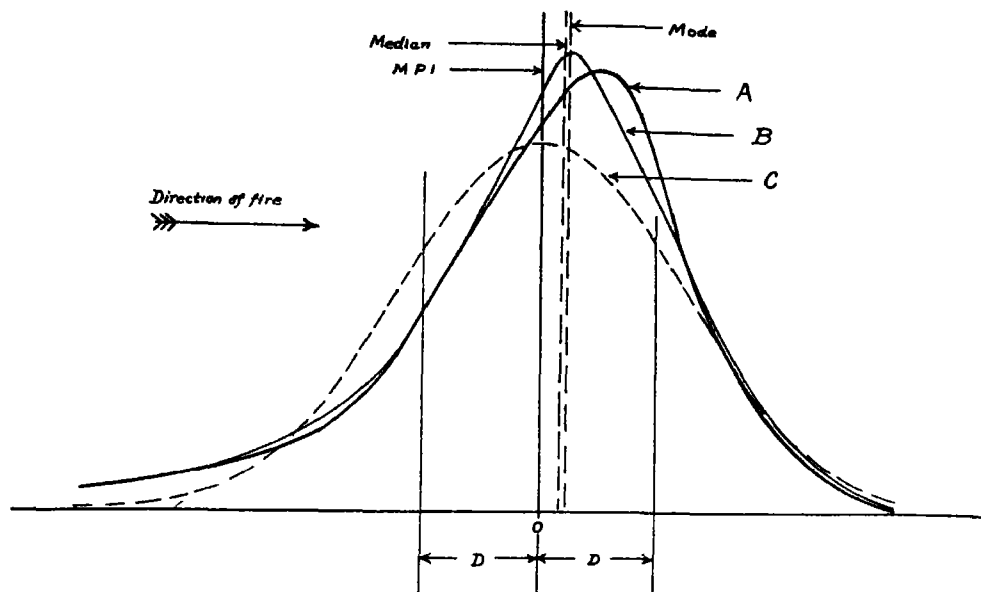


FIGURE 48

that their performances would better than others substantiate theory. On the contrary, the guns of these ships belong to a group whose patterns have been characterized by more than average dissymmetry. The analysis therefore represents conditions leaning toward the unfavorable for a test of the normal probability law.

In Figure 48, curve A represents the distribution of all 134 shots, considered as one large salvo with M.P.I. at O ; wild shots were *not* excluded. We see that this curve is again markedly lopsided and has the same general characteristics as curve A of Figure 47. But in this case it is illustrated more clearly how the greatest density of impacts occurs beyond the M.P.I. The *mode* for this shot distribution was determined as described in the previous illustration, and curve B is a plot of the shot distribution with respect to the *mode* as axis. The *median* is also shown. Curve B is practically symmetrical about its axis (the *mode*) except at the ends of its branches; while the right branch meets the X-axis at a distance of about $3.5D$ from the M.P.I., the left branch continues to remain above the X-axis well beyond the distance $4D$ from the M.P.I. (see last paragraph of article 1413). This indicates that in the main part of the pattern the distribution is symmetrical but that there are far more large "shorts" than large "overs." The distance between the M.P.I. and *mode* is about one-fourth of the mean dispersion, and the *median* is near the *mode*.

Illustration of
a range distri-
bution from
fleet firing

Curve *C*, Figure 48, shows the normal probability curve corresponding to the same data, i.e., to the same mean dispersion *D*. The discrepancies between curve *C* and the other curves are apparent, but let us investigate the essential differences. Curve *C* is *lower* and *flatter* than the others, therefore it actually represents greater mean dispersion than is indicated in the others (see article 1410). If a smaller value of *D* were used in plotting curve *C*, the peak of this curve would be extended upward and it would more nearly agree with the others.

The value of mean dispersion having been based on all the shots, *including wild shots*, would naturally be too great. After 12 of the 134 shots had been rejected

Comparison of the theoretical range distribution and the actual range distribution, corresponding to an actual case of fleet firing, excluding wild shots

as "wild,"* the actual distribution curves of the remaining 122 shots were plotted as shown in Figure 49, curve *A* being again the distribution with respect to the M.P.I., and curve *B* the distribution with respect to the *mode*. With the mean dispersion obtained from the remaining 122 shots, the theoretical distribution curve (or normal probability curve) was plotted and is shown as curve *C*. There is now very marked agreement among all three curves, the only difference worthy of note being the small distance between the axis of symmetry of *A* and *B*,

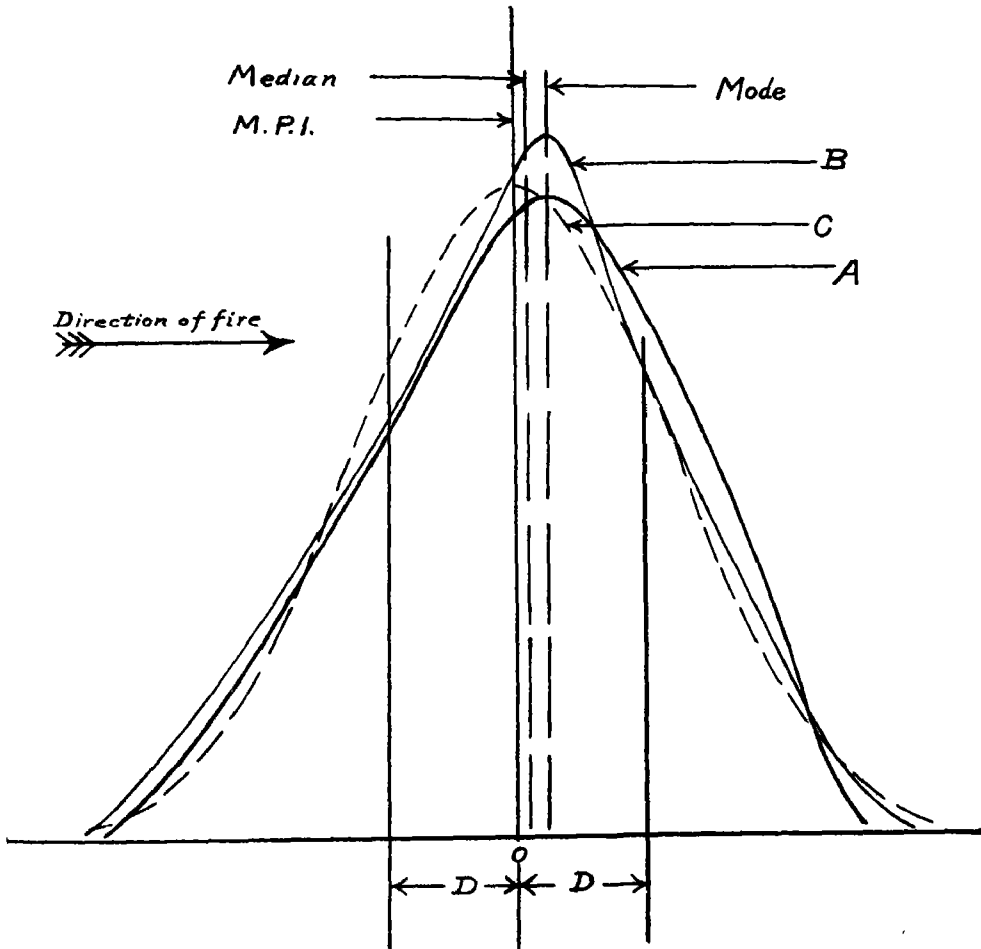


FIGURE 49

* The rejection of these 12 shots was made in accordance with a rule that will be discussed later (article 1503).

and that of C . This merely indicates that the densest portion of the impacts occurred a little beyond their M.P.I. (about $\frac{1}{2}D$), while the impacts taken altogether were distributed practically symmetrically.

The above analysis indicates very close agreement indeed between theory and practice, for somewhat more than 91% of the firing of three different ships throughout an entire battle practice. The ships selected were such as to indicate, if anything, less than the average chance of agreeing with theory. The conditions were actual service conditions and the errors included all errors that one may expect to find, both in the guns themselves and in their control. Of particular importance is the feature of wild shots. The above analysis indicates very strongly that with proper identification and rejection of wild shots, the fall of shot even at the longest of ranges follows the normal probability law. From inspection of Figure 49, which represents 122 out of 134 shots actually fired (or over 91%), it is apparent that percentage of hits by practice and by theory are very nearly the same, whence it appears that after rejecting wild shots we may expect the probability table to give us pretty good results "in the long run." In this case the regular procedure would have to be modified only by applying the probable percentage of hits to the probable percentage of good shots, e.g., if for a certain target we had a 10% chance of hitting, we would now call it $10\% \times 91\%$, getting finally about 9%.

The above analyses seem to justify the conclusion that the normal probability law applies only approximately to gunfire when we make no rejections of wild shots, but that it applies very well to gunfire when we reject wild shots on the basis of some judicious rule. No mention has been made in all this discussion of deflection distributions, for these are almost invariably symmetrical and follow the normal law even more closely than do range distributions.*

EXERCISES

1. *Given:* The points of fall of the shots of a 10-gun salvo were observed as follows, distances having been measured from the center of the target's waterline.

1.	375 yards short	30 yards right
2.	300 " short	10 " left
3.	250 " short	20 " left
4.	200 " short	0
5.	150 " short	10 " right
6.	50 " short	15 " right
7.	25 " short	40 " left
8.	0	5 " left
9.	100 " over	25 " left
10.	200 " over	35 " left

The hitting space of the target was 84 yards.

- Find:* (a) The location of the M.P.I. with respect to the center of the target's waterline.
 (b) The error of the M.P.I.
 (c) The pattern in range and in deflection.

* Some comparisons between theoretical and practical deflection distributions are given in articles 132 and 148, *Exterior Ballistics*, 1908, Alger. One of these is an analysis of 500 shots and shows theoretical and actual deflection distributions to agree within about 2% for errors from $1/4D$ to $3D$. Comparisons between theory and actual observations, and pertaining to observations of widely varying natures, may be found in articles 19 and 24, *The Combination of Observations*, Brunt.

- (d) The apparent mean dispersion in range and in deflection.
 (e) The true mean dispersion in range and in deflection.

Answers: (a) 105 yards short, 8 yards left
 (b) 147 yards short, 8 yards left
 (c) 575 yards in range, 70 yards in deflection
 (d) 150 yards in range, 18 yards in deflection
 (e) 158 yards in range, 19 yards in deflection

2. *Given:* The points of fall of an 8-gun salvo from a ship carrying 16"2600 f.s guns, were observed as follows, distances having been measured from the center of the target's waterline.

1.	325 yards over	18 yards right
2.	300 " "	14 " "
3.	225 " "	12 " "
4.	175 " "	6 " "
5.	125 " "	9 " left
6.	50 " "	10 " "
7.	0	12 " "
8.	75 " short	13 " "

The target was 36 feet high, 130 feet long, and had a depth† of 80 feet in the line of fire. The range was 20,000 yards.

- Find:* (a) The location of the M.P.I. with respect to center of the target's waterline.
 (b) The error of the M.P.I.
 (c) The pattern in range and in deflection.
 (d) The apparent mean dispersion in range and deflection.
 (e) The true mean dispersion in range and deflection.

Answers: (a) 141 yards over, and 1 yard right.
 (b) 107 yards over, and 1 yard right.
 (c) 400 yards in range, 31 yards in deflection.
 (d) 116 yards in range, and 12 yards in deflection.
 (e) 124 yards in range, and 13 yards in deflection.

3. *Given:* The type of gun, the horizontal range, the true mean dispersions* in range and in deflection, and the height of the target and its depth in and width across the line of fire (feet).

Find: The percentage of hits to be expected under the conditions given, and with M.P.I. at center of hitting space.

	Given							Answers
	Gun	Range (yds.)	True mean dispersion* (yds.)		Target (feet)			% Hits
			Range	Deflection	†Height	‡Depth	Width	
A	3"2700 f.s.	3500	25	3	6	30	60	41.0%
B	4"2900 f.s.	4000	16	1	15	0	18	98.0%
C	5"3150 f.s.	13,000	70	10	30	210	42	17.3%
D	12"2900 f.s.	20,000	130	15	40	90	600	17.7%
E	16"2600 f.s.	38,000	210	25	40	90	600	6.4%

* See art. 1418, par. 1. † Use Col. 7 of the range table. ‡ See art. 1402 (h).

4. *Given:* The type of gun, the horizontal range, the true mean dispersions* in range and in deflection, the height of the target and its depth in and width across the line of fire (feet), and error of M.P.I. as indicated.

Find: The percentage of hits to be expected under the conditions given. (Compare with Exercise 3.)

Given										Answers
	Gun	Range (yds.)	True mean dispersion* (yds.)		Target (feet)			Error of M.P.I. (yds.)		% Hits
			Range	Deflection	†Height	‡Depth	Width	Range	Deflection	
A	3"2700 f.s.	3500	25	3	6	30	60	50	10	6.5%
B	4"2900 f.s.	4000	16	1	15	0	18	18	2	77.2%
C	5"3150 f.s.	13,000	70	10	30	210	42	200	10	1.2%
D	12"2900 f.s.	20,000	130	15	40	90	600	300	50	3.3%
E	16"2800 f.s.	38,000	210	25	40	90	600	500	100	0.5%

5. *Given:* The type of gun, the horizontal range, the true mean dispersion* in range, and the height of the target and its depth in the line of fire, and the number of guns fired in salvo.

Find: The number of *shorts* that should be seen when the salvo pattern is centered on the target's hitting space.

Given							Answers
	Gun	Range (yds.)	No. shots in salvo	True mean dispersion* (yds.)	Target (feet)		No. Shorts
					†Height	‡Depth	
A	3"2700 f.s.	3500	4	25	6	30	1
B	4"2900 f.s.	4000	3	16	15	0	0
C	5"3150 f.s.	13,000	7	70	30	210	2
D	12"2900 f.s.	20,000	12	130	40	90	5
E	16"2800 f.s.	38,000	8	210	40	90	4

6. *Given:* A destroyer is 300 feet long and has a 30-foot beam. At battle ranges the hitting space due to the freeboard of the destroyer averages about 30 yards, for secondary battery guns. It is assumed that the true mean dispersion of secondary battery guns averages about 150 yards in range and about 25 yards in deflection at battle ranges, and that at these ranges the average error of M.P.I. for the same batteries is about 300 yards in range and about 50 yards in deflection (see article 1418, par. 1).

Find: How do the destroyer's chances of being hit compare under the following two conditions, (a) when she is steaming directly in the line of fire (target angle 0° or 180°), (b) when she is steaming directly across the line of fire (target angle 90° or 270°)?

Answers: (a) 0.3%.
(b) 1.2%.

* See art. 1418, par. 1. † Use Col. 7 of the range table. ‡ See art. 1402 (h).

CHAPTER 15

FURTHER APPLICATIONS OF THE LAWS OF PROBABILITY IN CONNECTION WITH SALVO ANALYSIS. DETERMINATION OF PROBABLE PATTERN SIZE, WILD SHOTS, AVERAGE SHIFT OF M.P.I., CONTROL ERRORS.

1501. The determination of the probable pattern size of a salvo is a problem which is beset with great difficulties, for it involves the determination of probabilities pertaining to a definite, small number of occurrences, whereas the laws of probability depend for their accuracy upon great numbers. Thus with the information, from the probability table, that there is about one chance in a thousand that an individual error of size $\pm 4D$ will occur, we can be reasonably certain that in a salvo of 1000 shots an error of this size *will* occur, and that the pattern size of such a salvo will approach $8D$. In a salvo of 10 shots there is still that one chance in a thousand that the error $\pm 4D$ will occur, but evidently it is likely to occur only about once in 100 such salvos. It is likely that the greatest errors occurring in most of the 10-gun salvos will be considerably less than $\pm 4D$, that the patterns of most of these salvos will be correspondingly smaller, and hence that the *average* pattern of these salvos will be smaller than $8D$.

A number of rules applicable to this problem have been devised by various authorities. One that shows the closest agreement with observed results in actual firings is given here. This rule is due to Mazzuoli and is based on the following considerations.* If $\pm a$ denotes the limiting dispersion of a salvo, and if there are 1000 shots in the salvo, the probability that just *one* of the 1000 shots will *fail* to be included within these limits, is evidently $P = .001$. Likewise the probability that *all but one* will be included, is $P = .999$. If $P = .999$, then $a/D = 4.0$, whence $\pm a = 4.0D$, and the pattern is $8D$. Now if n shots compose the salvo, the probability that all but one of them will be included within the limits $\pm a$, is evidently,

$$P = \frac{n-1}{n} \quad (1501)$$

as, for instance, with 100 shots the probability of all but one falling within the limits $\pm a$ is 99/100, and for 12 shots it is 11/12, etc. Having determined P as above, the value of a/D corresponding to this value of P will then define the limiting error, plus or minus, beyond which *probably* not more than *one* shot will fall. Since the error so found is always plus or minus, the corresponding pattern size will be equal to twice this error.

For a 12-gun salvo, (1501) gives $P = \frac{12-1}{12} = .917$; entering the probability table with this value of P , we find the corresponding value $a/D = 2.17$, whence $\pm a = 2.17D$, and the pattern size is $2 \times 2.17D = 4.34D$. Applying the same process

* Ref: Page 393, *Handbook of Ballistics*, Cranz and Becker.

to salvos of other sizes, we get the following ratios of pattern size to true mean dispersion. These ratios are slightly greater than those found in practice, but provide very good approximations.

	Number of shots in salvo	Ratio of pattern size to true mean dispersion
Ratio of pattern size to mean dispersion	3	2.43
	4	2.89
	5	3.21
	6	3.47
	7	3.67
	8	3.85
	9	4.00
	10	4.13
	11	4.24
	12	4.34

1502. Knowledge of the ratio of pattern size to mean dispersion affords a convenient method for determining approximately the probable percentage of hits. The average pattern size of a series of firings may be determined very readily from rake observations, it being necessary to measure only the distance between the shot farthest over and the shot farthest short in each salvo, and to take the average of these distances for the several salvos observed. Applying to this average pattern the ratio corresponding to the average number of shots of the salvos, we get the approximate mean dispersion.

An example will illustrate the application of the above features.

Given: A ship carrying a main battery of 12 guns has on several target practices had an average pattern of 1000 yards at battle ranges.

Find: The probable percentage of hits in range for this battery at battle ranges, on targets whose total hitting space averages 60 yards, with the M.P.I. at the center of the hitting space. What would it be if the pattern size were decreased to 600 yards?

Percentage of hits, based on average pattern size

For the 1000-yard pattern, $D_r = \frac{1000}{4.34} = 230$ yards.

For the 600-yard pattern, $D_r = \frac{600}{4.34} = 138$ yards.

In either case, $a_r = \frac{S'}{2} = \frac{60}{2} = 30$ yards.

For the 1000-yard pattern, $\frac{a_r}{D_r} = \frac{30}{230} = .13$, whence $P_r = .083$.

For the 600-yard pattern, $\frac{a_r}{D_r} = \frac{30}{138} = .22$, whence $P_r = .139$.

Thus with 1000-yard patterns the ship would expect a maximum of about 8.3% hits in range, and with 600-yard patterns about 13.9% hits in range. (These percentages are, of course, based on M.P.I. at center of hitting space and will be considerably reduced if the ship has large errors in control. They indicate, however, approximately the *best* the ship can hope to do under the stated conditions.)

1503. A problem similar in some aspects to that of determining the ratio of pattern size to mean dispersion, is that of determining how great the dispersion of a single shot must be before it may be classed as *wild*. The difficulty in establishing a sound criterion for determining what constitutes a *wild* result is by no means unique to the problem of gunfire. It exists throughout the entire field to which the laws of probability are applied and has received close attention by many of the world's foremost mathematicians.

The similarity between the problems of determining pattern size and of determining wild shots will be seen if we return to the discussion of how Mazzuoli's rule was derived (art. 1501, par. 2). We dealt there with the limiting error within which the shots of the salvo would *probably* be included; twice this limiting error gave us the pattern size. Now if this pattern size *probably* includes all shots, then any shots which actually fall outside of such a pattern may be thought of as *improbable* results, i.e., *wild shots*.

Mazzuoli's rule actually accounts for only $(n-1)$ shots of a salvo of n shots, and may therefore appear to be insufficiently rigid. The fact that it actually proves, on the contrary, to be somewhat too rigid when applied to the determination of pattern size (as evidenced by practical results), may indicate that our natural intuition in thinking of it as not rigid enough, is at fault. Mazzuoli himself stated that the rule serves well for determining wild shots, and its application to our own problems only serves to strengthen that conviction. The rejection of wild shots in connection with the analysis made in article 1427 and resulting finally in the curves shown in Figure 49, was made according to Mazzuoli's criterion.

From the discussion in the preceding paragraph it appears that a wild shot, according to Mazzuoli's criterion, is one which falls beyond the limits of the probable pattern size defined by the same criterion. Thus for a 12-gun salvo the probable pattern size is $4.34D$, which includes errors as great as $2.17D$ either short of or beyond the M.P.I., (i.e., $a = \pm 2.17 D$). This then defines as a wild shot for a 12-gun salvo one whose individual dispersion exceeds 2.17 times the mean dispersion of that salvo. The table in article 1501 can be used conveniently for determining wild shots according to this criterion, since one-half the ratio of pattern size to true mean dispersion defines the size of error which, if exceeded by a shot of that salvo, classifies that shot as wild.

1504. The above rule for determining wild shots is not offered here as a standard rule. Due to wide divergence of opinion on the subject, nothing approaching a "standard" rule exists. In our own service, practice has of late varied widely in this connection. For purposes of standardizing target practice reports, definite rules for determining wild shots are laid down in *Orders for Gunnery Exercises*; these rules must be followed in submitting target practice reports and are, for that purpose, "standard."

Proving grounds favor a criterion somewhat more rigid than Mazzuoli's; it is known as Chauvenet's criterion. The latter criterion is derived by the same reasoning as applied in deriving Mazzuoli's, but Chauvenet excludes but one-half of a shot of the n in a salvo, or series. The probability ratio is thus defined by the relation,

Chauvenet's
criterion for
rejection of
wild shots

whence,

$$P = \frac{n - \frac{1}{2}}{n}$$

$$P = \frac{2n - 1}{2n} \quad (1502)$$

in place of the relation given by (1501). Applying this to a 12-gun salvo we get, $P = \frac{24-1}{24} = \frac{23}{24} = .958$, whence $\frac{\sigma}{D} = 2.55$, and $\pm a = 2.55D$. According to this criterion a *wild* shot, for a 12-gun salvo, is then one whose dispersion exceeds 2.55 times the mean dispersion of that salvo, rather than 2.17 times, as gotten from Mazzuoli's criterion. Whichever of these rules is accepted, it will vary very considerably from the rule of calling *wild* only those shots whose dispersions exceed four times the mean dispersion, which is sometimes employed. The use of the latter rule may cause results to appear to be much more discordant than they actually are.*

1505. If a number of small salvos were fired in succession, each with the same point of aim and range and deflection settings, and if each impact and the M.P.I. of each salvo were plotted, we could consider the several impacts taken altogether as an aggregate salvo, and we could find the M.P.I. of this aggregate salvo in the usual manner. We would now find the M.P.I.s of the several small salvos grouped about the M.P.I. of the aggregate salvo, just as the individual impacts of each salvo would be grouped about their own M.P.I.s. This is explained by the occasional occurrence of large individual dispersions, which carry with them the M.P.I.s of the salvos on which they occur. Another evidence of this movement of the M.P.I.s is given by the fact that the pattern of the aggregate salvo would be considerably larger than that of each small salvo, whence it appears that the small patterns must shift about within the limits comprising the aggregate pattern.

Treating the M.P.I.s of the several small salvos as single impacts and measuring their mean dispersion in the usual manner, we now have what may be termed the *mean dispersion of the M.P.I.* Knowledge of how much this mean dispersion of the M.P.I. may amount to is of importance to the spotter, for it is an error over which he has no possible control. To attempt to correct for this natural movement of the M.P.I. by spots, is as illogical as to attempt to spot the individual shots of a salvo each to the salvo's M.P.I., and would prove as fruitless.

It is shown in the laws of errors that when the arithmetical mean of a number of observations is taken, this arithmetical mean is subject to an error which depends upon the number of observations entering into it; also that if D is the mean error of the individual observations from their arithmetical mean, D_0 the mean error of the arithmetical mean, and n the number of observation, then,

$$D_0 = \frac{D}{\sqrt{n}}. \quad (1503)^\dagger$$

This relation is directly applicable to our case if we call D_0 the mean dispersion of the M.P.I., D the mean dispersion of the gun, and n the number of shots in the salvo.

It is to be noted that the value of D_0 derived from formula (1503) represents the mean error of M.P.I. that is likely to occur due solely to the mean dispersion of the guns themselves and entirely independently of errors in control. Let us suppose, for example, that the true mean dispersion in range of a battery of six guns

* Ref: §65, *Handbook of Ballistics*, Cranz and Becker. In addition to the two mentioned above, three other rules are discussed in this reference, and the limits assigned by these also are very considerably less than $4D$. Ref. also Chapter VIII, *The Combination of Observations*, Brunt.

† Ref: Page 385, *Handbook of Ballistics*, Cranz and Becker. Also, article 101, *The Calculus of Observations*, Whitaker and Robinson. Also, article 21, *The Combination of Observations*, Brunt.

is 100 yards. Now if an unlimited number of salvos were to be fired from this battery with absolutely no differences in control from salvo to salvo, it would nevertheless be likely that the M.P.I. of any one of the 6-gun salvos would be in

error, with respect to the M.P.I. of *all* the shots, by an amount equal to $\frac{100}{\sqrt{6}} = 41$ yards.

1506. Of more immediate importance to the spotter, however, is knowledge as to the probable *shifts* of the M.P.I. between successive salvos. The amount of the probable shift of M.P.I. from salvo to salvo can be determined by means of the method of successive differences.* In this method the mean dispersion is determined by measuring the differences between successive observations in the order in which they actually occur, rather than by referring each observation to the arithmetical mean of all the observations. Thus in a series of shots the differences between the first and second shots, between the second and third, third and fourth, etc., are determined, and the sum of all these differences is divided by the number of differences. If E_d is the *average difference* between successive shots, as thus determined, the mean dispersion is then given by the relation

Method of
determining
mean disper-
sion from
successive
differences

$$D = \frac{E_d}{\sqrt{2}} \quad (1504)$$

Determination
of average
shift of M.P.I.

It will be noted that (1504) gives the relation between the mean dispersion and the *average difference between successive shots of a series*. We may apply this also to successive salvos, thus getting a relation between the mean dispersion of the M.P.I. and the average difference, or *shift*, between successive positions of the M.P.I. Thus, if we let D_{0d} denote the average *shift* of the M.P.I., and D_0 the *mean dispersion* of the M.P.I., we have,

$$D_0 = \frac{D_{0d}}{\sqrt{2}}$$

whence,

$$D_{0d} = D_0\sqrt{2}. \quad (1505)$$

For the case cited in the foregoing article, we have $D = 100$ yards, whence, from

(1503), $D_0 = \frac{100}{\sqrt{6}} = 41$ yards, and, from (1505), $D_{0d} = 41\sqrt{2} = 58$ yards. Thus we find that the *average shift of M.P.I.* is 58 yards, which is $\frac{58}{347}$, or practically $\frac{1}{6}$,

of the pattern size.

Applying this process to salvos of from 3 to 12 shots, we get the following values for the *average shift of M.P.I. between successive salvos, due only to the errors of the gun* (i.e., with no errors in control).

* Ref: §62, *Handbook of Ballistics*, Cranz and Becker.

Average shift
of M.P.I.,
due only to
gun errors,
in terms of
pattern size

Number of shots in salvo	Average shift of M.P.I. between successive salvos, due only to gun errors
3	$\frac{1}{3}$ Pattern
4	$\frac{1}{4}$ Pattern
5	$\frac{1}{5}$ Pattern
6	$\frac{1}{6}$ Pattern
7	$\frac{1}{7}$ Pattern
8	$\frac{1}{8}$ Pattern
9	$\frac{1}{9}$ Pattern
10	$\frac{1}{10}$ Pattern
11	$\frac{1}{11}$ Pattern
12	$\frac{1}{12}$ Pattern

The ratios in the right-hand column, all in terms of pattern size, are given to the nearest simple fraction.

1507. In actual practice it is found that the mean dispersion of the M.P.I. is greater than accounted for by formula (1503), and hence greater than indicated in the above table. The reason for this is the presence of control errors. In speaking of control errors, it is necessary to distinguish between those that influence the mean dispersion of the guns of the battery and those that operate equally upon the battery as a whole. In a sense, control errors may be considered to include the various sight-setting and pointing errors at pointer fire, or gun-laying errors at director fire, etc. These errors, however, can hardly be separated from the errors of the guns themselves; they are reflected in the mean dispersion of the guns and in the pattern of the battery. The shift of M.P.I. accounted for by (1503) will include these errors and we may consider them as part of the gun errors, rather than as control errors.

But errors in range-keeping, transmission of data to the guns, and, in the case of director fire, director-pointing errors, etc., affect the battery as a whole. These errors will not be reflected in increased pattern sizes, i.e., in increased dispersion among the guns, but in increased dispersion of the M.P.I.s of the salvos fired by these guns. The mean dispersion of M.P.I. that is actually observed is therefore composed of that portion which results from the gun errors themselves and is found from (1503), and of a portion due to the control of the battery as a whole. We will consider the term *control error* to refer to the latter portion.

It follows, now, that the average *shift* of M.P.I. to be expected in practice depends not altogether upon the mean dispersion of M.P.I. due to gun errors, but upon the mean dispersion of M.P.I. due to both gun and control errors. The values given in the above table therefore represent only a minimum. How much these values will be increased depends upon the size of the control errors to be expected. It remains true, however, that the *average shift of M.P.I.* is equal to $\sqrt{2}$ times the *mean dispersion of the M.P.I.*, and hence, by determining from target practice results what mean dispersion of M.P.I. is to be expected, we may determine what average *shift* of M.P.I. is to be expected. An example will illustrate these features.

Given: A ship carrying a 12-gun battery finds from target practice analysis that the average pattern size is 1000 yards and the mean dispersion of the M.P.I. is 100 yards.

Find: What average shift of M.P.I. is to be expected, in terms of pattern size? What would it be if the mean dispersion of the M.P.I. were 200 yards?

We are given, $D_0 = 100$ yards, and $D_0 = 200$ yards. From (1505) we find,

$$\text{and,} \quad D_{0_d} = 100\sqrt{2} = 141 \text{ yards}$$

$$D_{0_d} = 200\sqrt{2} = 282 \text{ yards.}$$

The ratios to pattern size then are,

$$\frac{141}{1000} = .141, \text{ or about } \frac{1}{7}$$

and

$$\frac{282}{1000} = .282, \text{ or about } \frac{1}{4}(+).$$

In the above example the 1000-yard pattern size indicated a mean dispersion of $\frac{1000}{4.34} = 230$ yards (article 1501). This mean dispersion of guns would cause a mean

dispersion of M.P.I. of $\frac{230}{\sqrt{12}} = 66$ yards (formula (1503)), and the average shift of

M.P.I. due to this would be $66\sqrt{2} = 93$ yards (formula (1505)), or $\frac{93}{1000} = \frac{1}{11}$ pat-

tern size, which agrees with the table in article 1506. The difference between this ratio and those found in the example will illustrate how materially the average shift of M.P.I. may be increased by the presence of control errors. *And it must be borne in mind that the spotter is not justified in applying corrections for control errors of the accidental type, any more than he is justified in applying corrections for the accidental errors of the gun.*

The application of a high degree of mathematical accuracy to the determination of such ratios as dealt with here, is hardly justifiable, in view of the very rough values with which we have to deal. Such figures as have been introduced in this discussion have been used principally to illustrate general conclusions. The table in article 1506 shows that even with perfect control the M.P.I. may shift considerably, and that the average shift for small salvos is a considerably greater proportion of the pattern size than is the case with large salvos. The example in this article shows that the average shift shown in the table may be increased materially by control errors. These considerations underlie the doctrine, followed by experienced spotters, regarding the least size of "centering spot" that they are justified in making; this is usually defined in terms of some simple fraction of the pattern size, as $\frac{1}{4}$ or $\frac{1}{2}$, etc. What this ratio should be for any typical set of conditions may be determined by the methods outlined above.

1508. Experienced spotters generally adhere to the rule that after the hitting range has once been established, a correction for a subsequent error of the M.P.I. should not be applied unless the error occurs at least twice in the same direction. It may be considered that this rule is dictated wholly by common sense, but as a matter of fact it is also well founded in the laws of probability. We know that the spotter must not attempt to apply corrections for the accidental errors of gunfire, which manifest themselves not only in the dispersion of the shots of a salvo but also in the dispersion of the M.P.I.s of successive salvos. We also know that it is

a characteristic of accidental errors that they are as likely to occur in one direction as in the opposite direction and that most likely they will alternate in direction. On the first occurrence of an error, i.e., after the hitting range has once been established, the spotter has no means of determining whether the error is one which should be corrected or whether it is an accidental error. If the error is of such size that it can be accounted for by the natural shifts of the M.P.I., the spotter has every reason to assume that it is indeed due to this cause and that a spot should therefore not be applied for it. But even if the error is much larger than can be accounted for by the natural shifts of the M.P.I., the spotter still is not justified in applying a spot for it on its first occurrence, for it may have been due, for example, to an error by the director pointer which will not be repeated. Now if the *large* error (i.e., the one last mentioned) should occur again, in the same direction, on the following salvo, there would be a strong probability that this error is due to a recurring cause; in this case the spotter would be fully justified in applying a correction for the error on its second occurrence. If the *smaller* error (i.e., the one first mentioned above) should occur again in the same direction on the following salvo, the odds would lie somewhat in favor of applying a spot on the second occurrence, but not so strongly as in the case of the larger error, for two successive *accidental* errors in the same direction can hardly be regarded as a highly improbable occurrence. For this reason some spotters prefer to withhold a spot for an error which is within the limits of the natural shifts of the M.P.I. until the error has occurred three times in the same direction.*

All of the above presumes, of course, that the spotter is entirely unaided by information from which he may infer that shifts of M.P.I. in a certain direction are likely to occur. If changes in conditions which he can observe, such as changes in target course, should indicate that a shift of M.P.I. in a certain direction is likely, the spotter should, by all means, at once apply a correction for an observed shift in that direction.

1509. The reduction of gun dispersion and of control dispersion are distinct problems, involving in the first case the design of good guns and the training of good gun crews, and in the other case the design of good range-keeping systems and the training of good fire-control parties. It is desirable, therefore, to have a means for separating into its component parts the total error that is observed. It would involve insurmountable difficulties to allocate to all the contributory sources their respective shares, but it is possible to make a reasonably good subdivision among the broader classifications.

One of the fundamental relations given in the theories of errors is that when an aggregate error is composed of separate and independent component errors, the aggregate error is equal to the square root of the sum of the squares of the component errors.† Thus, for our case, if we let the observed mean dispersion of the M.P.I. (D_0) be composed of a portion D_g due to the gun, and a portion D_c due to the control, we have,

$$D_0 = \sqrt{D_g^2 + D_c^2} \quad (1506)$$

whence we get also,

* The probability that the error will in any one instance occur in one of two opposite directions is $\frac{1}{2}$. The probability that it will occur n successive times in the same direction is $(\frac{1}{2})^n$. Hence there is one chance in four that it will occur twice in the same direction and only one chance in eight that it will occur three times in the same direction. Ref. §25 and §52, *Probability and Its Engineering Uses*, Fry.

† Ref: Articles 15 and 26, *The Combination of Observations*, Brunt. Also, p. 381, *Handbook of Ballistics*, Crans and Becker.

$$D_0 = \sqrt{D_0^2 - D_g^2} \quad (1507)$$

Now if we analyze a target practice and find a certain mean dispersion for the guns and also a certain mean dispersion of M.P.I., formula (1507) enables us to estimate how great the mean dispersion in control was. An example will best illustrate this.

Given: From target practice results it was determined that the mean dispersion in range of the guns of a certain main battery was 200 yards. The mean dispersion in range of the M.P.I. was found to be 150 yards. The average number of shots per salvo was 11.

Find: The mean dispersion in control.

We are given $D = 200$ yards, and from (1503) we get,

$$D_g^* = \frac{200}{\sqrt{11}} = 60 \text{ yards.}$$

We are given, also, $D_0 = 150$ yards. Substituting in (1507) we get

$$D_0 = \sqrt{150^2 - 60^2} = 137 \text{ yards.}$$

Whence it appears that the mean dispersion in control was 137 yards.

By means of the same principle we may determine the size of the errors entering into the mean dispersion of the guns on board ship and not accounted for by the mean dispersion measured at the proving ground. For instance:

**Determination
of components
of mean dis-
persion due to
shipboard
use of guns**

Given: The mean dispersion of a certain gun as measured at the proving ground was found to be 100 yards at a certain range. On board ship a battery of these guns gives a mean dispersion of 150 yards at the same range.

Find: What is the size of the errors incident to the use of the guns on board ship, at that range?

We shall let D_s denote the error to be found. The aggregate error being composed of the proving-ground component (100 yards) and the ship component (D_s), the square root of the sum of the squares of these must equal the aggregate error (150 yards). Whence,

$$\sqrt{D_s^2 + 100^2} = 150$$

or

$$D_s = \sqrt{150^2 - 100^2} = 112 \text{ yards}$$

The result shows that a mean error of 112 yards is caused by factors entering into the use of the guns on board ship, which includes variations in ammunition and loading, pointers' errors, etc., as well as differences among the various guns of the battery (see also article 1406).

1510. Probability of hitting, i.e., the expected percentage of hits, is often thought of only in terms of the mean dispersion of the guns. Percentage of hits deduced only on the basis of gun dispersion is not a true measure of the battle efficiency of a ship or of a fleet. The percentage of hits to be expected depends also upon the distance of the M.P.I. from the center of the hitting space, i.e., upon the *mean error of the M.P.I.* The mean error of the M.P.I. might be found by taking the square root of the sum of the squares of all the separate errors entering into it. This would be a highly theoretical value unless we knew accurately the values of all the separate errors. Actually

* If the value of D used in formula (1503) is the mean dispersion of the gun, the resulting value of D_g will be the mean dispersion of M.P.I. due only to the gun, or D_{g_0} .

it is far more practical to measure the value of the mean error of the M.P.I. from target practice results, just as we measure gun dispersion by actually firing the gun, rather than by trying to deduce it theoretically. For the purpose of studying the individual causes of the error of the M.P.I., we may break it down into such components as we desire to study (for instance as noted in article 1509). But as far as determination of the probability of hitting is concerned, we are interested only in what the mean error actually amounts to. No other measure of this will be better than that obtained from actual practice.

Target practice reports include a record of the mean error of the M.P.I. from the center of the hitting space, for each ship at each long range practice. These quantities may be averaged over a period of years for a particular ship, or by classes of ships, or even for an entire fleet, depending upon the nature of the problem with which we may be concerned. The quantity so obtained will represent, for the particular ship, class of ships, or fleet, respectively, the success which may be expected in placing the M.P.I. on the target. Let us consider, for instance, a problem of the following type. It is desired to compare the probable effects of fire against ships which are steaming broadside to the line of fire (i.e., with target angles of 90° or 270°), and the probable effects of fire against ships which are steaming in the line of fire (i.e., with target angles of 0° or 180°). The answers at which we would arrive would vary considerably depending upon whether we considered the M.P.I. to be at the center of the hitting space, or considered it to be at some average distance from that point. Evidently the true answer would be that which would be arrived at by using a typical mean error of M.P.I., as well as a typical mean dispersion.

The methods of solving such problems as these are illustrated in article 1420, and the problems in article 1421 are of the nature just discussed here. In these problems we found 2.2% to be the probable percentage of hits on a certain target steaming *broadside* to the line of fire, and 0.8% the probable percentage of hits on the same target steaming *in* the line of fire, the M.P.I. having in both cases been considered to be a certain distance away from the center of the hitting space. In article 1418 (problems 2 and 3) the same conditions were assumed, except that the M.P.I. was considered to be at the center of the hitting space; the results were respectively, 5.7% and 3.2%. While the values of mean dispersion and error of M.P.I. assumed were not necessarily typical, the results of these solutions show how misleading the results may be if we neglect, in any study of this nature, to consider the error of the M.P.I.

EXERCISES

1. *Given*: The type of gun, the horizontal range, the number of guns usually fired in salvo, the average pattern in range,* and the average hitting space for the given gun at the given range against its usual target.

Find: The maximum percentage of hits to be expected from the given battery under the given conditions.

	Given					Answers
	Gun	Range (yds.)	Size of Salvo	Average Pattern* (yds.)	Hitting Space (yds.)	% Hits
A	3"2700 f.s.	3000	4	250	180	58.9%
B	4"2900 f.s.	5000	3	250	180	51.2%
C	5"3150 f.s.	8000	7	400	120	33.9%
D	12"2900 f.s.	20,000	12	1000	60	8.3%
E	16"2800 f.s.	25,000	8	700	70	12.1%

* See article 1418, par. 1.

APPENDIX A*

THE PRACTICAL APPLICATION OF SIACCI'S METHOD TO THE SOLUTION OF THE TRAJECTORY. INGALLS' METHOD.

New Symbols Introduced.†

- $a, b, a', b', t', z, \dots$ The general form of Ingalls' secondary functions, pertaining to elements at any point in the trajectory (art. 1611).
- $A, B, A', B', T', Z, \dots$ The special form of Ingalls' secondary functions, pertaining to terminal elements (art. 1611).
- $a_0, b_0, a_0', b_0', t_0', z_0, \dots$ The special form of Ingalls' secondary functions, pertaining to elements at the vertex (art. 1611).
- H, \dots A special secondary function used only for finding the maximum ordinate (art. 1615).
- Q, \dots A special secondary function used in connection with the solution for the coefficient of form (art. 1621).

1601. It is to be expected that with the completion of Volume III of the War Department Tables (art. 714), Siacci's Method will be abandoned altogether. In the meantime Siacci's Method continues to be the most convenient method for solving the trajectory for angles of departure not exceeding 15° . The following outline covers the principal features entering into the practical application of Siacci's Method; extracts from the tables required therewith are given at the end of this appendix. A more complete treatment of Siacci's Method may be found in Chapters 5-11 of the 1926 and 1930 editions of *Exterior Ballistics*, and the complete tables required therewith in the 1926 and 1930 editions of *Range and Ballistic Tables*.‡

1602. We have already seen in article 510 how Siacci's range formula was simplified to the form

$$x = C_s(S_u - S_v) \quad (519) \quad (1601)$$

in which S_u , which is called the *space function*, is a tabulated function representing the integral

The space function
$$S_u = -\frac{1}{A} \int \frac{du}{u^{(\alpha-1)}} \quad (518) \quad (1602)$$

Proceeding in similar fashion, other formulas are set up as follows.

1603. The solution of (515) can be simplified by tabulating the values of an *inclination function* I_u which represents the integral

The inclination function
$$I_u = -\frac{2g}{A} \int \frac{du}{u^{(\alpha+1)}} \quad (1603)$$

* It is desirable that the study of this appendix be preceded by a review of Chapter 5.

† The functions A and α introduced here are not to be confused with Mayevski's constant A and exponent α (Chapter 4).

‡ The 1926 and 1930 editions of *Exterior Ballistics* and *Range and Ballistic Tables* are carried in most ships' libraries.

Formula (515), after integration, then becomes

$$\tan \phi - \tan \theta = \frac{C_s}{2 \cos^2 \phi} (-I_V + I_u)$$

whence

$$\tan \theta = \tan \phi - \frac{C_s}{2 \cos^2 \phi} (I_u - I_V). \quad (1604)$$

1604. Substituting in (504) the value of v given by (508), we have

$$dt = - \frac{u \cos \phi}{g} \sec^2 \theta d\theta$$

and substituting in the above the value of $\sec^2 \theta d\theta$ given by (514)

$$dt = - \frac{u \cos \phi}{g} \times \frac{C_s g}{A \cos^2 \phi} \times \frac{du}{u^{(a+1)}}$$

which reduces to

$$dt = - \frac{C_s}{A \cos \phi} \times \frac{du}{u^a}.$$

The integration of the above is expressed as follows

$$\int_0^t dt = - \frac{C_s}{A \cos \phi} \int_V^u \frac{du}{u^a}. \quad (1605)$$

The solution of (1605) is simplified by tabulating the values of a *time function* T_u which represents the integral

$$\text{The time function} \quad T_u = - \frac{1}{A} \int \frac{du}{u^a}.$$

Formula (1605), after integration, then becomes

$$t = \frac{C_s}{\cos \phi} (T_u - T_V). \quad (1606)$$

1605. Formula (1604) can be re-written

$$\tan \theta - \tan \phi = - \frac{C_s}{2 \cos^2 \phi} (I_u - I_V).$$

Multiplying the above by formula (516) we have

$$dx \tan \theta - dx \tan \phi = \frac{C_s}{2 \cos^2 \phi} (I_u - I_V) \times \frac{C_s}{A} \times \frac{du}{u^{(a-1)}}.$$

Since $dx \tan \theta = dy$ we may make this substitution in the above, and we may further rearrange it as follows,

$$dy - dx \tan \phi = \frac{C_s^2}{2 \cos^2 \phi} \left[I_u \left(\frac{1}{A} \times \frac{du}{u^{(a-1)}} \right) - I_V \left(\frac{1}{A} \times \frac{du}{u^{(a-1)}} \right) \right].$$

The integration of the above is expressed as follows,

$$\int_y^0 dy - \int_x^0 dx \tan \phi = \frac{C_s^2}{2 \cos^2 \phi} \left[\frac{1}{A} \int_u^V I_u \times \frac{du}{u^{(a-1)}} - I_V \times \frac{1}{A} \int_u^V \frac{du}{u^{(a-1)}} \right]. \quad (1607)$$

The solution of (1607) is simplified by tabulating the values of an *altitude function* A_u which represents the integral

$$\text{The altitude function} \quad A_u = - \frac{1}{A} \int I_u \times \frac{du}{u^{(a-1)}}. \quad (1608)$$

It will be observed that A_u is the negative of the quantity which appears in (1607) as the left-hand member within the bracket; also that the right-hand member within the bracket is equal to $I_V \times S_u$. Formula (1607), after integration, then becomes

$$y - x \tan \phi = - \frac{C_s^2}{2 \cos^2 \phi} [A_u - A_V - I_V(S_u - S_V)]. \quad (1609)$$

It is to be noted that I_V , which is a definite value of I for the definite value $u = V$, is handled as a constant in the integration of the right-hand member within the bracket; I_u , on the other hand, is a variable value of I and hence enters into the integration.

Dividing (1609) by (1601), and transposing $\tan \phi$, we have

$$\frac{y}{x} = \tan \phi - \frac{C_s}{2 \cos^2 \phi} \left[\frac{A_u - A_V}{S_u - S_V} - I_V \right]. \quad (1610)$$

1606. We have now found a series of expressions which give us useful relations between certain elements of the trajectory, and they are grouped here for convenience:

$$x = C_s(S_u - S_V) \quad (1601)$$

$$\tan \theta = \tan \phi - \frac{C_s}{2 \cos^2 \phi} (I_u - I_V) \quad (1604)$$

Siacci's
general
ballistic
formulas

$$t = \frac{C_s}{\cos \phi} (T_u - T_V) \quad (1606)$$

$$\frac{y}{x} = \tan \phi - \frac{C_s}{2 \cos^2 \phi} \left[\frac{A_u - A_V}{S_u - S_V} - I_V \right] \quad (1610)$$

For finding the value of v corresponding to a given value of u we may rewrite (508),

$$v = u \cos \phi \sec \theta \quad (1611)$$

The above equations are *Siacci's ballistic formulas* and they are the fundamental equations upon which the solutions by Siacci's Method are based.

1607. The values of the *inclination, time, and altitude functions* have been tabulated in the manner already indicated in article 510 for the *space function*.

Siacci prepared tables of these functions, but the tables used in the **Ingalls' Table I** United States were prepared by Ingalls and are generally known as *Ingalls' Table I*; the latter table includes values of S_u , A_u , I_u , and T_u for velocities from 3600 f.s. to 100 f.s.*

* Further details connected with the tabulation of these functions are given on pp. 80-83, *Exterior Ballistics, 1926*, and pp. 75-78, *Exterior Ballistics, 1930*.

1608. The ballistic formulas listed in article 1606 are expressions for finding the various elements at *any point* in the trajectory. In order to apply them to the point of fall we proceed as follows.

For the point of fall we have,

$$x=X, \quad y=0, \quad t=T, \quad v=v_w, \quad u=u_w, \quad \theta=-\omega$$

Making the substitutions in (1601) we have

$$X = C_s (S_{u_w} - S_V)$$

whence

$$S_{u_w} = S_V + \frac{X}{C_s} \quad (1612)$$

From (1610) we have,

$$\frac{0}{X} = \tan \phi - \frac{C_s}{2 \cos^2 \phi} \left[\frac{A_{u_w} - A_V}{S_{u_w} - S_V} - I_V \right]$$

whence

$$\tan \phi = \frac{C_s}{2 \cos^2 \phi} \left[\frac{A_{u_w} - A_V}{S_{u_w} - S_V} - I_V \right] \quad (1613A)$$

and since

$$2 \cos^2 \phi \tan \phi = \frac{2 \cos^2 \phi \sin \phi}{\cos \phi} = \sin 2\phi$$

we have finally,

$$\sin 2\phi = C_s \left[\frac{A_{u_w} - A_V}{S_{u_w} - S_V} - I_V \right] \quad (1613)$$

From (1604) we have,

$$\tan (-\omega) = \tan \phi - \frac{C_s}{2 \cos^2 \phi} [I_{u_w} - I_V]$$

Putting for $\tan \phi$ its value from (1613A) we have,

Siacci's
ballistic
formulas
for
terminal
elements

$$\tan (-\omega) = \frac{C_s}{2 \cos^2 \phi} \left[\frac{A_{u_w} - A_V}{S_{u_w} - S_V} - I_V \right] - \frac{C_s}{2 \cos^2 \phi} [I_{u_w} - I_V]$$

which simplifies to

$$\tan \omega = \frac{C_s}{2 \cos^2 \phi} \left[I_{u_w} - \frac{A_{u_w} - A_V}{S_{u_w} - S_V} \right] \quad (1614)$$

From (1606) we have,

$$T = C_s \sec \phi (T_{u_w} - T_V) \quad (1615)$$

From (1611) we have,

$$v_w = u_w \cos \phi \sec \omega \quad (1616)$$

1609. We have in formulas (1612) to (1616) a series of expressions from which the desired terminal elements may be found as follows. Let us suppose that it is

desired to determine the angle of departure that is required for a given range and initial velocity, and also the angle of fall, time of flight, and striking velocity corresponding to the given conditions. X and V are then the given quantities; C can be found as usual from δ , w , i , and d , but the value of C , which is required for the solution of the formulas depends upon the values of f_a and β (art. 509), and the latter depend, respectively, on the maximum ordinate and angle of departure (arts. 507 and 512), neither of which is known. To overcome this difficulty we may proceed as follows.

Knowing V we find S_V from *Ingalls' Table I*, and using this in (1612), with C in place of C_0 , we get a first approximation of S_{u_0} , and reentering *Ingalls' Table I* with this value of S_{u_0} we find u_0 . Knowing u_0 and V we now find from *Ingalls' Table I* the values of A , S , and I required for the solution of (1613), and, still using C in place of C_0 , we find from (1613) a first approximation of ϕ . Proceeding similarly, we get from (1615) a first approximation of T ; with the latter we can find an approximate value of the maximum ordinate from

$$y_* = \frac{gT^2}{8}. \quad (313\ C)$$

With the first approximation of ϕ we can now get a first approximation of β from the relation (see art. 512)

$$\beta = \sqrt{\sec \phi}$$

and using $\frac{2}{3}$ of the approximate value of y_* (see art. 507) we can get a first approximation of f_a from a table in which f_a is tabulated against altitude (an extract from this table is given in art. 1617). The first approximations of β and f_a are now used to get a first approximation of C , by means of formula (513), and the entire procedure outlined above is repeated, and second approximations of β , f_a , and C , are obtained. Further approximations are made until two successive values of C , agree within the chosen limits of accuracy.

The solution of formulas (1612) to (1616) is then completed as follows. Entering *Ingalls' Table I* with the given V , the values of S_V , A_V , I_V , and T_V are found. With the given value of X , the value of C , found by successive approximations, and the value of S_V found from the table, formula (1612) is solved for S_{u_0} . Entering the table with the latter, u_0 , A_{u_0} , I_{u_0} , and T_{u_0} are found, and formulas (1613), (1614), (1615), and (1616) are solved for ϕ , ω , T , and v_0 , respectively.

1610. The procedure outlined in the preceding article constitutes a comparatively direct solution of the trajectory by means of Siacci's principles, and it was at one time employed in the computation of U. S. Navy range tables.† This procedure is, however, a laborious one, not only because of the many values that must be taken from *Ingalls' Table I* and of the tedious interpolations involved therein, but also because the formulas themselves are not in convenient form for solution by logarithms.

* Although this formula is based on the trajectory in vacuum, it gives a fairly good value of the maximum ordinate of the trajectory in air when the value of T used in it is the time of flight in air. However, a somewhat more accurate determination of the maximum ordinate is rendered possible by a further development of Siacci's Method, which will be outlined presently.

† Ref. Chapter VII, *Exterior Ballistics*, 1906, and Chapter 8, *Exterior Ballistics*, 1915, by Professor P. R. Alger.

The idea of further simplifying Siacci's Method by tabulating values of the frequently recurring combinations of the functions S_u , A_u , I_u , and T_u , was suggested as early as 1883 by Braccialini. The development of this idea to a point of practical usefulness is due to Ingalls, and the greatly simplified method resulting from Ingalls' further treatment of Siacci's Method is therefore commonly referred to as *Ingalls' Method*, or as the *Ingalls-Siacci Method*.

1611. Ingalls created a number of *secondary functions* which represent combinations of the *primary functions* S_u , A_u , I_u , and T_u . In the form in which they are applicable to elements at *any* point in the trajectory, these secondary functions are defined by the relations

$$a = \frac{A_u - A_v}{S_u - S_v} - I_v \quad (1617)$$

$$b = I_u - \frac{A_u - A_v}{S_u - S_v} \quad (1618)$$

Ingalls'
secondary
functions for
elements at any
point in the
trajectory

$$a' = a + b = I_u - I_v \quad (1619)$$

$$t' = T_u - T_v \quad (1620)$$

$$b' = \frac{b}{a} \quad (1621)$$

$$z = \frac{x}{C_s} \quad (1622)$$

In the form in which they are applicable to the *terminal elements*, the same secondary functions are defined by the relations

$$A = \frac{A_{u_w} - A_v}{S_{u_w} - S_v} - I_v \quad (1617A)$$

$$B = I_{u_w} - \frac{A_{u_w} - A_v}{S_{u_w} - S_v} \quad (1618A)$$

Ingalls'
secondary
functions
for terminal
elements

$$A' = A + B = I_{u_w} - I_v \quad (1619A)$$

$$T' = T_{u_w} - T_v \quad (1620A)$$

$$B' = \frac{B}{A} \quad (1621A)$$

$$Z = \frac{X}{C_s} \quad (1622A)$$

Special forms of the same secondary functions may be denoted by special subscripts. For example, at the vertex, where u is denoted by u_s , we have

$$b_s = I_{u_s} - \frac{A_{u_s} - A_v}{S_{u_s} - S_v} \quad (1618B)$$

$$a_s' = a_s + b_s = I_{u_s} - I_v \quad (1619B)$$

and others similarly.

It is to be noted that the secondary functions do not involve any new factors, but that they merely represent combinations of the primary functions. The Siacci C (i.e., C_s) is to be used with the secondary functions just as with the primary functions.

1612. The secondary functions are introduced into Siacci's ballistic formulas (art. 1606) as follows.

From (1601) we have,

$$S_u = S_v + \frac{x}{C_s}$$

whence

$$S_u = S_v + z \quad (1623)$$

From (1613) we may write directly,

$$\sin 2\phi = AC_s \quad (1624)$$

for it will be noted that the expression in the bracket is identical with that given for A in (1617A).

From (1604) we may write,

$$\tan \theta = \tan \phi - \frac{C_s a}{2 \cos^2 \phi} \quad (1625)$$

and from (1610)

$$\frac{y}{x} = \tan \phi - \frac{C_s a}{2 \cos^2 \phi} \quad (1626)$$

From (1625) we may write

$$\tan \theta = \tan \phi \left(1 - \frac{a' C_s}{2 \cos^2 \phi \tan \phi} \right)$$

or

$$\tan \theta = \tan \phi \left(1 - \frac{a' C_s}{2 \cos^2 \phi} \times \frac{\cos \phi}{\sin \phi} \right)$$

whence

$$\tan \theta = \tan \phi \left(1 - \frac{a' C_s}{\sin 2\phi} \right) \quad (1627)$$

From (1626), by the same process, we get

$$\frac{y}{x} = \tan \phi \left(1 - \frac{a C_s}{\sin 2\phi} \right) \quad (1628)$$

Putting in (1628) the value of $\sin 2\phi$ from (1624) we have,

$$\frac{y}{x} = \tan \phi \left(1 - \frac{a C_s}{A C_s} \right)$$

or

$$\frac{y}{x} = \tan \phi \left(1 - \frac{a}{A} \right)$$

whence

$$\frac{y}{x} = \frac{\tan \phi}{A} (A - a)$$

or

$$y = \frac{\tan \phi}{A} (A - a)x \quad (1629)$$

From (1627), by the same process, we get,

$$\tan \theta = \frac{\tan \phi}{A} (A - a') \quad (1630A)$$

and since $a' = a + b$

$$\tan \theta = \frac{\tan \phi}{A} [A - (a + b)]$$

or

$$\tan \theta = \frac{\tan \phi}{A} \left[A - a \left(1 + \frac{b}{a} \right) \right]$$

and since $b' = b/a$

$$\tan \theta = \frac{\tan \phi}{A} \left[A - a(1 + b') \right] \quad (1630)*$$

From (1606) we may write directly,

$$= Cx' \sec \phi \quad (1631)$$

1613. We now have the following formulas from which the elements at *any* point in the trajectory can be found.

$$y = \frac{\tan \phi}{A} (A - a)x \quad (1629)$$

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formulas

$$\tan \theta = \frac{\tan \phi}{A} \left[A - a(1 + b') \right] \quad (1630)$$

$$t = C_x t' \sec \phi \quad (1631)$$

$$v = u \cos \phi \sec \theta \quad (1611)$$

1614. In order to solve the above formulas we must know the values of the secondary functions. We have already seen in article 1611 that the secondary functions are merely combinations of the primary functions, and it will be apparent that if we have enough information to find the values of the primary functions we also have enough information to find the values of the secondary functions. The tabulation of the secondary functions was performed by Ingalls, and the tables containing them are generally known as *Ingalls' Table II*.

An important consideration which enters into the tabulation of the secondary functions is the fact that these functions pertain to an integration between two velocities both of which depend upon the immediate problem at hand. *Ingalls' Table II* is therefore actually a series of tables each based on a certain value of V . In the complete *Ingalls' Table II* there are forty-nine separate tables for as many different values of V , ranging from 825 f.s. to 3600 f.s. The initial velocity V is thus one of the arguments on which *Ingalls' Table II* is based. There must be one more argument, for V pertains only to the initial conditions of the problem. The other argument must be an element pertaining to the point to be found. The element chosen for this purpose is the range x , or abscissa, of the point to be found. It is necessary, however, to base the range argument on certain standards, and this is done by applying to it the ballistic coefficient. The second argument on

which *Ingalls' Table II* is based is therefore $\frac{x}{C_x}$, and we have seen in article 1611

that this relation is represented by the secondary function z .

We have seen in article 1611 that z pertains to *any* point in the trajectory, Z to the point of fall, and z_v to the vertex. Consequently if we are dealing with elements at *any* point (x, y) the entering arguments for *Ingalls' Table II* are V

* Although formula (1630A) is actually in more convenient form than formula (1630), the latter renders it unnecessary to tabulate the function a' .

and $z = \frac{x}{C_s}$; if we are dealing with terminal elements the arguments are V and

$Z = \frac{X}{C_s}$; if we are dealing with vertex elements the arguments are V and $z_s = \frac{x_s}{C_s}$.

Likewise, if we enter the table with z we will find a, b', t' , etc.; if we enter with Z we will find A, B', T' , etc.; and if we enter with z_s we will find a_s, b_s', t_s' , etc.

1615. The solution for the terminal elements and maximum ordinate is further simplified as follows.

From (1613) and (1617A) we get directly,

$$\sin 2\phi = AC_s. \quad (1632)$$

From (1614) and (1618A) we get,

$$\tan \omega = \frac{BC_s}{2\cos^2 \phi}$$

or

$$\tan \omega = \frac{BC_s}{2\cos^2 \phi} \times \frac{\sin \phi}{\sin \phi}$$

whence, since $2 \sin \phi \cos \phi = \sin 2\phi$, and since $\frac{\sin \phi}{\cos \phi} = \tan \phi$, we may write,

$$\tan \omega = \frac{BC_s}{\sin 2\phi} \tan \phi.$$

Substituting for $\sin 2\phi$ its value AC_s from (1632) we have,

$$\tan \omega = \frac{BC_s}{AC_s} \tan \phi$$

and remembering from (1621A) that $B/A = B'$ we have, finally

$$\tan \omega = B' \tan \phi \quad (1633)$$

From (1615) and (1620A) we have,

$$T = C_s T' \sec \phi \quad (1634)$$

From (1616) we have,

$$v_\omega = u_\omega \cos \phi \sec \omega. \quad (1635)$$

In article 1612 we have derived the formula

$$\tan \theta = \frac{\tan \phi}{A} (A - a') \quad (1630 A)$$

which pertains to *any* point in the trajectory. If we apply this formula to the vertex, we have $\tan \theta = 0$ and $a' = a_s'$, while $\tan \phi$ and A are not affected, for ϕ depends only on the initial conditions and A depends on X which is the total range. Rewriting (1630 A) for the vertex we have

$$0 = \frac{\tan \phi}{A} (A - a_s').$$

The quantity $\tan \phi/A$ cannot equal zero, for both ϕ and A have finite values. It follows that $(A - a_s')$ must equal zero, and hence we have

$$A - a_s' = 0$$

or

$$A = a_s'.$$

This means, then, that the a' -function for the vertex, which is a_s' , is equal to the a -function for the point of fall, which is A . This is a fortunate coincidence, for it means that if A is known a_s' also is known, and hence z_s and all other secondary functions pertaining to the vertex can be found. However, we are interested principally in the maximum ordinate, and in simplifying the procedure for finding this element.

By substituting a_s' for A in (1629), the latter may then be written for the vertex as follows

$$y_s = \frac{\tan \phi}{a_s'} (a_s' - a_s) x_s.$$

Now from (1619 B) we have the relation $a_s' = a_s + b_s$, whence $b_s = a_s' - a_s$. Making this substitution in the above we get

$$y_s = \frac{b_s x_s}{a_s'} \tan \phi. \quad (1636)$$

We have already seen that if we know A we also know a_s' and hence can find

z_s and b_s ; knowing z_s we can find x_s from the relation $z_s = \frac{x_s}{C_s}$. In order to find A ,

however, we must know Z , and in order to find Z we must know X ; the solution of (1636) therefore must be based on a given value of X . The latter can be introduced into (1636) as follows

$$y_s = \left[\frac{b_s x_s}{a_s' X} \right] X \tan \phi. \quad (1637)$$

The secondary
function H

making it possible to compute the quantity in the bracket for any value of X and to tabulate it. The quantity in the bracket thus defines another secondary function, which is denoted by H , whence we have

$$H = \frac{b_s x_s}{a_s' X}. \quad (1638)$$

Substituting H in (1637) we have

$$y_s = H X \tan \phi. \quad (1639)$$

The function H is found from *Ingalls' Table II* in exactly the same manner in which the functions for the terminal elements are found. That is to say, although H pertains to the vertex it is not to be found with z_s as argument as is the case with other vertex functions (art. 1614). H is, in fact, a combination of vertex functions which, for convenience, has been tabulated in terms of the same argument which pertains to the terminal elements, i.e., $Z = \frac{X}{C_s}$.

1616. We now have the following formulas from which the terminal elements and the maximum ordinate can be found.

$$\sin 2\phi = AC_s. \quad (1632)$$

$$\tan \omega = B' \tan \phi \quad (1633)$$

$$T = C_s T' \sec \phi \quad (1634)$$

$$v_\omega = u_\omega \cos \phi \sec \omega \quad (1635)$$

$$y_s = H X \tan \phi. \quad (1639)$$

Ingalls'
ballistic
formulas
for terminal
elements and
maximum
ordinate

Let us suppose that our problem is to find the angle of departure that is required for a given range and initial velocity, and also the angle of fall, time of flight, striking velocity, and maximum ordinate corresponding to the given conditions. This is what we have to do in constructing a range table. V and X are known; from δ , w , i , and d we can compute C . With X and C we can get a first approximation of Z from formula (1622A). With the given value of V and with the first approximation of Z we now enter *Ingalls' Table II* and find first approximations of the secondary functions A and H . With the latter we solve (1632) and (1639) and thus obtain the first approximations of ϕ and y , and thence the first approximations of β , f_a , and C_s , as already outlined in article 1609. With the first approximation of C_s we find second approximations of Z , A , and H , and thence of ϕ and β and of y , and f_a , and eventually of C_s . Further successive approximations are made until two successive values of C_s agree within the chosen limits of accuracy. The final value of Z is then computed by means of formula (1622A), the corresponding values of A , B' , T' , u_w , and H are found from *Ingalls' Table II*, and formulas 1632-1635, and 1639, are solved for ϕ , ω , T , v_w , and y , respectively.

1617. The practical application of Ingalls' formulas to the solution of trajectories is illustrated in the following problems. The values of the secondary functions required in connection with these problems may be found in the extracts from *Ingalls' Table II* appended to this chapter; these extracts represent about one one-hundredth of the complete table, which includes values of the argument Z from 0 to 20,000 for each of forty-nine values of the argument V .*

The values of f_a required in connection with these problems may be found from the following table, which is an extract from the *altitude-factor table* used with Ingalls' Method; although the complete table includes values of the argument h up to 40,000 feet, the maximum value of h that occurs within the limits of angle of departure to which Ingalls' Method is now restricted, is about 4000 feet.*

$$h = \frac{2}{3}y.$$

Altitude-factor table	h (ft.)	f_a	Δ	h (ft.)	f_a	Δ	h (ft.)	f_a	Δ
	0	1.0000	.0030	1000	1.0300	.0032	2000	1.0620	.0032
	100	1.0030	.0030	1100	1.0332	.0032	2100	1.0652	.0032
	200	1.0060	.0030	1200	1.0364	.0032	2200	1.0684	.0032
	300	1.0090	.0030	1300	1.0396	.0032	2300	1.0716	.0032
	400	1.0120	.0030	1400	1.0428	.0032	2400	1.0748	.0032
	500	1.0150	.0030	1500	1.0460	.0032	2500	1.0780	.0032
	600	1.0180	.0030	1600	1.0492	.0032	2600	1.0812	.0032
	700	1.0210	.0030	1700	1.0524	.0032	2700	1.0844	.0032
	800	1.0240	.0030	1800	1.0556	.0032	2800	1.0876	.0032
	900	1.0270	.0030	1900	1.0588	.0032	2900	1.0908	.0032
	1000	1.0300	.0032	2000	1.0620	.0032	3000	1.0940	.0034

It will be recalled (art. 507) that the altitude factor f_a , as used in Siacci's Method, accounts for the variation of atmospheric density with altitude and is based on a mean density for the entire trajectory; and that the practice in our services, in this connection, is to assume for this mean density the density at an altitude equal to two-thirds of the maximum ordinate of the trajectory. The value of f_a is therefore to be found from the above table by entering the latter with a value of h which is equal to two-thirds of the maximum ordinate, i.e., with $h = 2/3y$.

* The complete *Ingalls' Table II* and *altitude-factor table* are given in *Artillery Circular "M"* (U. S. War Department, 1917). Extracts of *Table II* covering the initial velocities of practically all U. S. naval guns, as well as the complete *altitude-factor table*, are given in the 1926 and 1930 editions of *Range and Ballistic Tables*.

1618. The following example illustrates the process of solving for range-table values of ϕ , ω , T , v_u , and y , corresponding to a given range, initial velocity, and projectile.

Solution for
other terminal
elements when
 X is known

Given: The 16"2600 f.s. gun, weight of projectile 2100 lbs., coefficient of form .61, horizontal range 10,000 yards, atmospheric conditions standard.

Find: The angle of departure, angle of fall, time of flight, striking velocity, and maximum ordinate corresponding to the given conditions.

The first step is to determine C , by successive approximations (see arts. 1609 and 1616).

First Approximation

$$C = \frac{w}{\delta i d^2} \quad (406)$$

$w=2100$	log 3.32222
$\delta=1.00$	colog 0.00000
$i=.61$	log 9.78533-10
$d^2=256$	colog 0.21467
C	colog 7.59176-10
	log 1.12865

$$Z_1 = \frac{X}{C} \quad (1622A)$$

$X=30,000$ feet.....	log 4.47712
C	colog 8.87135-10
$Z_1=2230.8$	log 3.34847

From Ingalls' Table II, with $V=2600$ and $Z=2230.8$

$$A_1 = .01259 + .308 \times .00069 = .012803$$

$$H_1 = .27322 + .308 \times .00115 = .27357$$

$$\sin 2\phi_1 = A_1 C \quad (1632) \qquad y_{s_1} = H_1 X \tan \phi_1 \quad (1639)$$

$A_1=.012803$	log 8.10731-10		
C	log 1.12865	$H_1=.27358$	log 9.43709-10
$2\phi_1=9^\circ 54' 51''$	log 9.23596-10	$X=30,000$ feet..	log 4.47712
$\phi_1=4^\circ 57' 26''$			ltan 8.93820-10
$y_{s_1}=711.8$ feet.....			log 2.85239

From altitude-factor table, entering with $\frac{2}{3}y_{s_1}=475$, $f_{s_1}=1.0142$

$$\beta_1 = \sqrt{\sec \phi_1}; 4^\circ 57' 26'', \text{lsec } .00162, \frac{1}{2} \text{ lsec } .00081$$

Second Approximation

$$C_{s_1} = \frac{f_{s_1}}{\beta_1} C \quad (513)$$

C	log 1.12865
$f_{s_1}=1.0142$	log .00613
β_1	colog 9.99919-10
C_{s_1}	log 1.13397

$$Z_1 = \frac{X}{C_{s_1}}$$

$X = 30,000$ feet.....	log 4.47712
C_{s_1}log 1.13397.....	colog 8.86603-10
$Z_1 = 2203.7$	log 3.34315

From Ingalls' Table II, with $V = 2600$ and $Z = 2203.7$

$$A_2 = .01259 + .037 \times .00069 = .012616$$

$$H_2 = .27322 + .037 \times .00115 = .27326$$

$$\sin 2\phi_2 = A_2 C_{s_1}$$

$$y_{s_1} = H_2 X \tan \phi_2$$

$A_2 = .012616$log 8.10092-10	
C_{s_1}log 1.13397.....	$H_2 = .27326$ log 9.43658-10
$2\phi_2 = 9^\circ 53' 22''$lsin 9.23489-10	$X = 30,000$ feet.. log 4.47712
$\phi_2 = 4^\circ 56' 41''$	ltan 8.93712-10
$y_{s_1} = 709.3$ feet.....	log 2.85082

From altitude-factor table, with $\frac{2}{3}y_{s_1} = 473$, $f_{s_1} = 1.0142$

$$\beta_2 = \sqrt{\sec \phi_2}; 4^\circ 56' 41'', \text{ lsec } .00162, \frac{1}{2} \text{ lsec } .00081$$

Third Approximation

$$C_{s_2} = \frac{f_{s_2}}{\beta_2} C$$

C	log 1.12865
$f_{s_2} = 1.0142$..	log .00613
β_2log .00081.....	colog 9.99919-10
C_{s_2}	log 1.13397

$$Z_2 = \frac{X}{C_{s_2}}$$

$X = 30,000$ feet.....	log 4.47712
C_{s_2}log 1.13397.....	colog 8.86603-10
$Z_2 = 2203.7$	log 3.34315

From Ingalls' Table II, with $V = 2600$ and $Z = 2203.7$

$$A_3 = .01259 + .037 \times .00069 = .012616$$

$$H_3 = .27322 + .037 \times .00115 = .27326$$

$$\sin 2\phi_3 = A_3 C_{s_2}$$

$$y_{s_2} = H_3 X \tan \phi_3$$

$A_3 = .012616$log 8.10092-10	
C_{s_2}log 1.13397.....	$H_3 = .27326$ log 9.43658-10
$2\phi_3 = 9^\circ 53' 22''$lsin 9.23489-10	$X = 30,000$ feet.. log 4.47712
$\phi_3 = 4^\circ 56' 41''$	ltan 8.93709-10
$y_{s_2} = 709.2$ feet.....	log 2.85079

From altitude-factor table, with $\frac{2}{3}y_{s_2} = 473$, $f_{s_2} = 1.0142$

$$\beta_3 = \sqrt{\sec \phi_3}; 4^\circ 56' 41'', \text{ lsec } .00162, \frac{1}{2} \text{ lsec } .00081$$

Fourth Approximation

$$C_{\bullet} = \frac{f_{\bullet}}{\beta_{\bullet}} C$$

It will be noted that $f_{\bullet} = f_{\bullet}$, and $\beta_{\bullet} = \beta_{\bullet}$, whence it follows that $C_{\bullet} = C_{\bullet}$; it is therefore unnecessary to continue with the fourth approximation. The final values of C_{\bullet} , Z , ϕ , and y_{\bullet} are those which have been arrived at in the third approximation, viz.,

$$\log C_{\bullet} = 1.13397, \quad Z = 2203.7, \quad \phi = 4^{\circ} 56' 41'', \quad y_{\bullet} = 709.2 \text{ feet.}$$

The solution is now completed as follows:

From Ingalls' Table II, with $V = 2600$ and $Z = 2203.7$

Log $B' = .07820 + .037 \times .00370 = .07834$		
$T' = .969 + .037 \times .051 = .97089$		
$u_{\omega} = 1984.4 - .037 \times 25.4 = 1983.5$		
$\tan \omega = B' \tan \phi,$ (1633)	$T = C_{\bullet} T' \sec \phi,$ (1634)	$v_{\omega} = u_{\omega} \cos \phi \sec \omega$ (1635)
$C_{\bullet} \dots \dots \dots \log 1.13397$		
$\phi = 4^{\circ} 56' 41'' \dots \dots \dots \text{ltan } 8.93709 - 10$	$\text{lsec } 0.00162$	$\text{lc} \cos 9.99838 - 10$
$B' \dots \dots \dots \log 0.07834$		
$\omega = 5^{\circ} 54' 57'' \dots \dots \dots \text{ltan } 9.01543 - 10$		$\text{lsec } 0.00232$
$T' = .97089 \dots \dots \dots \log 9.98717 - 10$		
$T = 13.267 \text{ sec.} \dots \dots \dots \log 1.12276$		
$u_{\omega} = 1983.5 \dots \dots \dots \log 3.29743$		
$v_{\omega} = 1986.7 \text{ f.s.} \dots \dots \dots \log 3.29813$		

1619. The method of solving for elements at a point other than the point of fall is illustrated by the following example.

**Solution for
elements at
any point x, y .**

Given: The terminal elements of the 10,000-yard trajectory of the 16''2600 f.s. gun, under standard atmospheric conditions, as already found in the example of art. 1618.

Find: The ordinate, angle of inclination, time of flight, and remaining velocity at the point whose abscissa is 2000 yards.

From the example of art. 1618 we have

$$\log C_{\bullet} = 1.13397, \quad Z = 2203.7 \quad \phi = 4^{\circ} 56' 41''.$$

From formula (1622) we have $z = \frac{x}{C_{\bullet}}$

$z = 6000 \text{ feet.} \dots \dots \dots \log 3.77815$	
$C_{\bullet} \dots \dots \dots \text{colog } 8.86603 - 10$	
$z = 440.74 \dots \dots \dots \log 2.64418$	

From Ingalls' Table II

With $V = 2600$ and $Z = 2203.7$

$$A = .01259 + .037 \times .00069 = .012616$$

With $V = 2600$ and $z = 440.74$

$$\log b' = .01380 + .4074 \times .00350 = .01523$$

$$t' = .158 + .4074 \times .041 = .17470$$

$$u = 2479.3 - .4074 \times 29.5 = 2467.3$$

$$y = \frac{\tan \phi}{A} (A - a)x, \quad (1629)$$

$$\tan \theta = \frac{\tan \phi}{A} [A - a(1 + b')], \quad (1630)$$

$$t = C_s t' \sec \phi, \quad (1631)$$

$$v = u \cos \phi \sec \theta \quad (1611)$$

$$\begin{array}{lll} A = .012616, & a = .0021778, & (A - a) = .010438 \\ \log b' = .01523, & b' = 1.0357, & (1 + b') = 2.0357 \end{array}$$

$$(1 + b') = 2.0357 \dots \log .30871$$

$$a = .0021778 \dots \log 7.33802 - 10$$

$$a(1 + b') = .0044333 \dots \log 7.64673 - 10$$

$$a(1 + b') = .0044333, \quad A = .012616, \quad [A - a(1 + b')] = .0081827$$

$$\phi = 4^\circ 56' 41'' \dots \tan 8.93709 - 10 \quad \tan 8.93709 - 10 \quad \sec .00162 \quad \sec 9.99838 - 10$$

$$A = .012616 \dots \csc 1.89908 \dots \csc 1.89908$$

$$(A - a) = .010438 \quad \log 8.01862 - 10$$

$$x = 6000 \text{ feet} \dots \log 3.77815$$

$$y = 429.5 \text{ feet} \dots \log 2.63294$$

$$[A - a(1 + b')] = .0081827 \dots \log 7.91290 - 10$$

$$\theta = 3^\circ 12' 42'' \dots \tan 8.74907 - 10 \quad \sec .00068$$

$$C_s \dots \log 1.13397$$

$$t' = .17470 \dots \log 9.24229 - 10$$

$$t = 2.3872 \text{ sec} \dots \log 0.37788$$

$$u = 2467.3 \dots \log 3.39222$$

$$v = 2462.0 \text{ f.s.} \dots \log 3.39128$$

1620. The following example illustrates the method of solution when ϕ is given instead of X (compare with the example in art. 1618).

Solution for
terminal
elements when
 ϕ is known

Given: The 16''2600 f.s. gun, weight of projectile 2100 lbs.,
coefficient of form .61, angle of departure $4^\circ 56' 41''$,
atmospheric conditions standard.

Find: The horizontal range and maximum ordinate.

As in the example of art. 1618, the first step is to determine C_s by successive approximations; in the present case, however, the successive approximations are required only for the purpose of determining f_s , for β can be found directly from the given value of ϕ .

Let us compute C and then apply β to it and call the result C_s .

First Approximation

$$C = \frac{w}{\delta d^2} \quad (406)$$

$$w = 2100 \dots \log 3.32222$$

$$\delta = 1.00 \dots \csc 0.00000$$

$$i = .61 \dots \log 9.78533 - 10 \quad \csc 0.21467$$

$$d^2 = 256 \dots \log 2.40824 \quad \csc 7.59176 - 10$$

$$C \dots \log 1.12865$$

$$C_{s_1} = \frac{1}{\beta} C$$

$$\beta = \sqrt{\sec \phi}, \phi = 4^\circ 56' 41'', \text{lsec. .00162, } \frac{1}{2} \text{lsec. .00081}$$

C	log 1.12865
β	log .00081
C_{s_1}	log 1.12784

From (1632) we get,

$$A_1 = \frac{\sin 2\phi}{C_{s_1}}$$

$\phi = 4^\circ 56' 41''$	lsin 9.23490-10
$2\phi = 9^\circ 53' 22''$	lsin 9.23490-10
C_{s_1}	log 1.12784
$A_1 = .012796$	log 8.10706-10

From Ingalls' Table II, with $V = 2600$ and $A = .012796$

$$Z_1 = 2200 + \frac{.000206}{.00069} \times 100 = 2229.9$$

$$H_1 = .27322 + \frac{.000206}{.00069} \times .00115 = .27356$$

$$X_1 = Z_1 C_{s_1} \quad (1622A) \quad y_{s_1} = H_1 X_1 \tan \phi \quad (1639)$$

$Z_1 = 2229.9$	log 3.34828
C_{s_1}	log 1.12784
$X_1 = 29,931$ feet	log 4.47612
$H_1 = .27356$	log 9.43706-10
$\phi = 4^\circ 56' 41''$	ltan 8.93709-10
$y_{s_1} = 708.4$ feet	log 2.85027

From altitude-factor table, with $\frac{2}{3}y_{s_1} = 472$, $f_{s_1} = 1.0142$

Second Approximation

$$C_s = \frac{f_{s_1}}{\beta} C \quad (513)$$

C	log 1.12865
$f_{s_1} = 1.0142$	log .00613
β	log .00081
C_{s_2}	log 1.13397

$$A_2 = \frac{\sin 2\phi}{C_{s_2}}$$

$2\phi = 9^\circ 53' 22''$	lsin 9.23490-10
C_{s_2}	log 1.13397
$A_2 = .012616$	log 8.10093-10

From Ingalls' Table II, with $V=2600$ and $A=.012616$

$$Z_1 = 2200 + \frac{.000026}{.00069} \times 100 = 2203.8$$

$$H_1 = .27322 + \frac{.000026}{.00069} \times .00115 = .27326$$

$$X_2 = Z_1 C_{s_1} \qquad y_{s_1} = H_1 X_2 \tan \phi$$

$Z_1 = 2203.8$	log 3.34317
C_{s_1}	log 1.13397
$X_2 = 30,001$ feet	log 4.47714
$H_1 = .27326$	log 9.43658
$\phi = 4^\circ 56' 41''$	ltan 8.93709
$y_{s_1} = 709.3$ feet	log 2.85081

From altitude-factor table, with $\frac{2}{3} y_{s_1} = 473$, $f_{s_1} = 1.0142$

Third Approximation

$$C_{s_2} = \frac{f_{s_2} C}{\beta}$$

C	log 1.12865
$f_{s_2} = 1.0142$	log .00613
β	log .00081
C_{s_2}	colog 9.99919 - 10
	log 1.13397

We now find that $C_{s_2} = C_{s_1}$. Since we have already used this final value of C , in the second approximation and since there would be no changes in any other quantities, X_2 and y_{s_1} , represent the final values of X and y .

1621. The considerations involved in the determination of the coefficient of form i from the results of experimental ranging have already been discussed in articles 624-626, and the practical features entering into the determination of i by means of numerical-integration ballistic tables have been treated in detail in articles 802-814. It will be recalled that the process used in connection with the numerical-integration tables is the equivalent of a solution backwards from a measured X in order to find the value of i which satisfies this known value of X . The process used with Ingalls' Method is essentially the same, although less direct. It involves, first of all, the use of an additional secondary function which is denoted by Q .

The function Q is actually the ratio between the ranges in vacuum and in air for conditions that are otherwise identical. It enters into the problem as follows. From formula (308 B) we have

$$\sin 2\phi = \frac{g}{V^2} \times X_{(\text{vacuum})} \qquad (1640)$$

and from (1632),

$$\sin 2\phi = AC,$$

whence, since $C_s = \frac{X_{(air)}}{Z}$, (1622A)

$$\sin 2\phi = \frac{A}{Z} \times X_{(air)} \quad (1641)$$

Since we are using the same conditions for both air and vacuum, $\sin 2\phi$ is the same in both (1640) and (1641); equating the right-hand members of (1640) and (1641) we have,

$$\frac{g}{V^2} \times X_{(vacuum)} = \frac{A}{Z} \times X_{(air)}$$

whence,

The function Q

$$\frac{X_{(vacuum)}}{X_{(air)}} = Q = \frac{AV^2}{gZ} \quad (1642)$$

With (1642) we may compute Q for any given values of V and Z , and tabulate it with the other functions on the proper page for V and in the proper line for Z . The quantity tabulated is $\log Q$. For instance, suppose that we have $V = 2600$ and $Z = 5000$; we find the corresponding value of A to be 0.03762. Then Q may be computed from (1642),

$$Q = \frac{.03762 \times (2600)^2}{32.16 \times 5000} = 1.5815$$

$$\log Q = .19907$$

which, we see, is the value tabulated in the 2600 f. s. table for $Z = 5000$.

It will be apparent, now, that if we know V and Q we may find the corresponding value of Z , just as we might find Z with V and any other function. In the experimental determination of i we use this feature. Q is the ratio between range in vacuum and range in air for conditions otherwise identical. If we fire a gun with a known elevation and velocity we can easily determine the corresponding range in vacuum from (308),

$$X_{(vacuum)} = \frac{V^2 \sin 2\phi}{g}$$

The value of $X_{(air)}$ is actually measured. The value of Q is found from the relation

$$Q = \frac{X_{(vacuum)}}{X_{(air)}} = \frac{V^2 \sin 2\phi}{gX_{(air)}} \quad (1643)$$

Ingalls' Table II is now entered with the measured V , and with $\log Q$ as found from (1643), and the corresponding value of Z is found. With the latter, C_s is determined from the relation

$$C_s = \frac{X_{(air)}}{Z}$$

and i is then deduced from the relations

$$C_s = \frac{f_a}{B} C \quad (513)$$

whence

$$C_s = \frac{f_s w}{\beta \delta i d^2}$$

and

$$i = \frac{f_s w}{\beta \delta C_s d^2}. \quad (1644)$$

1622. The following is an example of the solution for i by Ingalls' Method.

Determina-
tion of i in
Ingalls'
Method

Given: In the experimental ranging of the 16''/2600 f.s. gun the following average results were obtained for a group of seven shots fired at angle of departure 8° : uncorrected observed range, 14,773 yards; observed initial velocity, 2580 f.s.; observed weight of projectiles 2102 lbs.; observed ballistic density, .962. The corrections to be applied to the observed range were found to be: for ballistic wind, $(-)$ 31 yards; for height of gun, $(-)$ 39 yards; for curvature of the earth, $(-)$ 87 yards.

Find: The value of i that corresponds to the given conditions.

The corrected observed range is

$$X = 14,773 - 31 - 39 - 87 = 14,616 \text{ yards} = 43,848 \text{ feet.}$$

$$Q = \frac{V^2 \sin 2\phi}{gX} \quad (1643)$$

$V = 2580 \text{ f.s.}$log 3.411622 log 6.82324
$\phi = 8^\circ, 2\phi = 16^\circ$	lsin 9.44034-10
$g = 32.16 \text{ f.s.s.}$log 1.50732colog 8.49268-10
$X = 43,848 \text{ feet}$log 4.64195colog 5.35805-10
Q	log 0.11431

From Ingalls' Table II, with $V = 2580$ and $\log Q = 0.11431$.

For 2500 f. s.

$$Z = 3000 + \frac{.00076}{.00417} \times 100 = 3018.2$$

$$H = .28326 + \frac{.00076}{.00417} \times .00123 = .28348$$

For 2600 f. s.

$$Z = 3000 + \frac{.00211}{.00411} \times 100 = 3051.3$$

$$H = .28254 + \frac{.00211}{.00411} \times .00121 = .28316$$

For 2580 f. s.

$$Z = 3018.2 + .80 \times 33.1 = 3044.7$$

$$H = .28348 - .80 \times .00032 = .28322$$

$$C_s = \frac{X}{Z}$$

$X = 43,848$ feet.....	log 4.64195
$Z = 3044.7$	log 3.48354.....colog 6.51646-10
C_s	log 1.15841

$$y_s = HX \tan \phi \quad (1639)$$

$H = .28322$	log 9.45212-10
$X = 43,848$ feet.....	log 4.64195
$\phi = 8^\circ$	ltan 9.14780-10
$y_s = 1745.3$ feet.....	log 3.24187

whence, from the altitude-factor table, with $2/3 y_s = 1164$, $f_s = 1.035$. Also

$$\beta = \sqrt{\sec 8^\circ}, \text{ or } \log \beta = 0.00212.$$

The solution is now completed as follows:

$$i = \frac{f_s w}{\beta \delta C_s d^2} \quad (1644)$$

$f_s = 1.035$	log 0.01494
$w = 2102$	log 3.32263
β	log 0.00212.....colog 9.99788-10
$\delta_b = .962$	log 9.98318-10.....colog 0.01682
C_s	log 1.15841.....colog 8.84159-10
$d^2 = 256$	log 2.40824.....colog 7.59176-10
$i = .61041$	log 9.78562-10

1623. The process illustrated in the foregoing example can also be used to determine the value of i that corresponds to range-table values. The following example will make this clear.

**Determination
of i correspond-
ing to range-
table values**

Given: In the 16''2600 f.s. range table the angle of departure and maximum ordinate for the 10,000-yard trajectory are, respectively, $4^\circ 56' 8$ and 709 feet.

Find: The value of i , according to Ingalls' Method, that corresponds to the given conditions.

$$Q = \frac{V^2 \sin 2\phi}{gX} \quad (1643)$$

$V = 2600$ f.s.....	log 3.41497.....	2 log 6.82994
$2\phi = 9^\circ 53' 6$		lsin 9.23506-10
$g = 32.16$ f.s.s.....	log 1.50732.....	colog 8.49268-10
$X = 30,000$ feet.....	log 4.47712.....	colog 5.52288-10
Q		log 0.08056

From Ingalls' Table II, with $V = 2600$ and $\log Q = 0.08056$

$$Z = 2200 + \frac{.00027}{.00390} \times 100 = 2206.9$$

$$C_s = \frac{X}{Z}$$

$X = 30,000$ feet.....	log 4.47712
$Z = 2206.9$	log 3.34378.....
	colog 6.65622-10
C_s	log 1.13334

From the altitude-factor table, with $2/3 y_s = 473$, $f_a = 1.0142$.

Also,

$$\beta = \sqrt{\sec 4^\circ 56' 8''}, \text{ or } \log \beta = 0.00081$$

$$i = \frac{f_a w}{\beta \delta C_s d^2} \quad (1644)$$

$f_a = 1.0142$	log 0.00613
$w = 2100$	log 3.32222
β	log 0.00081.....
	colog 9.99919-10
$\delta = 1.00$	log 0.00000.....
	colog 0.00000
C_s	log 1.13334.....
	colog 8.86666-10
$d^2 = 256$	log 2.40824.....
	colog 7.59176-10
$i = .61088$	log 9.78596-10

APPENDIX B

THE LIMITATIONS OF COLUMNS 7 AND 19 AT VERY SHORT RANGES. DETERMINATION OF HITTING SPACE AT VERY SHORT RANGES.

1701. It is a common error to make direct use of Column 7 of the range table for finding how far beyond a target the point of fall of a shot should occur if the shot pierces the vertical plane of the target at a given height above the water. Column 7 is not designed to give this information and at very short ranges large errors result from using it directly for this purpose. This is illustrated by the following example. Let us suppose that in Figure 50 OMH represents the 2500-yard trajectory of the 16''2600 f.s. gun, and that TM is a 20-foot target so located

Limitations of
Column 7 for
determining
hitting space
at very short
ranges

that the trajectory OMH just touches the top of the target. By definition (art. 914), the distance TH is the *danger space* for the 2500-yard trajectory OMH and the 20-foot target TM . From Column 7 of the 16''2600 f.s. range table we find that at the range 2500 yards the danger space of a 20-foot target is equal to 387 yards.

In Figure 50 we then have $OH = 2500$ yards, $TH = 387$ yards, and $OT = 2500 - 387 = 2113$ yards. It is to be noted, however, that the distance $TH = 387$ yards was found from Column 7 of the range table by entering the

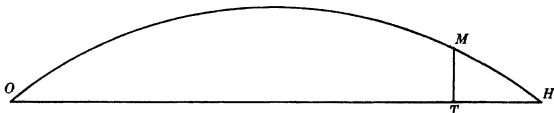


FIGURE 50

latter with 2500 yards, which is the distance from the gun to the point of fall of the trajectory that just touches the top of the target, and not by entering the range table with 2113 yards, which is the distance from the gun to the target. If we should seek to find the distance TH from Column 7 by entering the range table with the *target distance* 2113 yards, we would find the value 489 yards, which we already know is not correct. By definition (art. 1402 (h)), the distance TH in Figure 50 is the *hitting space* for the target TM at the *target distance* OT . For the case illustrated, $TH = 387$ yards is then the hitting space for the 20-foot target at the *target distance* 2113 yards, as compared to the danger space of 489 yards which is found from Column 7 for the same target at the same *target distance*.

The problem of determining the relation between points of impact in the vertical plane of the target and on the surface of the water enters into the construction of spotting diagrams. We have seen above that, with respect to a given *target distance*, this relation is defined by the quantity known as the *hitting space* rather than by the quantity known as the *danger space*, and also that at the short

range considered these quantities differ materially. It can be shown that for ranges exceeding about 5000 yards the distinction between danger space and hitting space ceases to have any practical significance; at such ranges Column 7 may be used directly for finding either of these quantities. The particular problem to be dealt with here is the development of a convenient method for determining the hitting space for ranges which occur at Short Range Practice (1600–2100 yards).

1702. One method of determining the hitting space is to deduce it from the danger space in the following manner. In Figure 50, TH is evidently the danger space at range OH and also the hitting space at range OT . Thus if S is the danger space at any given range X , then $S' = S$ is the hitting space at the range $X - S$.* However, the determination of the hitting space from this relation involves an indirect and inconvenient process. Moreover, the values appearing in Column 7 of many of the range tables still in use at the time of this writing were computed by a now obsolete method (see note at bottom of page 119); at short ranges the value of danger space found by this method may be in error by more than 25%, and any values of hitting space deduced from the danger space are subject to the errors of the latter. These undesirable features of the indirect method outlined above are avoided in the following approximate solution for the hitting space which is direct, simple, and sufficiently accurate for any practical purpose.

1703. In Figure 51 a gun is located at O and a target of height h is located at the range $OH = X$. The hitting space is $S' = HH'$.

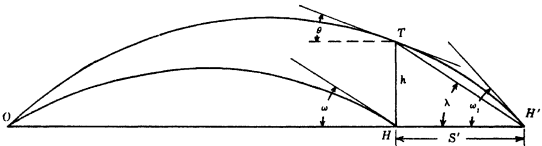


FIGURE 51

The value of S' is given exactly by the relation $S' = h \cot \lambda$, in which λ is the angle $TH'H$. This angle is not a known quantity, but it will be observed (Figure 51) that it lies between the angle of fall ω_1 of the trajectory OTH' which just touches the top of the target, and the angle of inclination θ of the same trajectory at the point T where the latter touches the top of the target. The angle of fall ω of the trajectory having the horizontal range OH also lies between the angles ω_1 and θ . Hence it appears that ω is probably very nearly equal to λ , and that the hitting

* This is, in fact, simply the reverse of the process now used for determining the danger space for short ranges (arts. 917–918). That is to say, the latter process actually determines the hitting space first and then deduces from it the danger space. There is no good reason for not tabulating hitting space in Column 7 in the first place, for at short ranges danger space is an altogether useless quantity, and at long ranges it is useful only by virtue of the fact that at such ranges it is practically equal to hitting space. But the fact that danger space is, nevertheless, tabulated in all of our range tables, makes it necessary to engage in this special study of how to convert it into a useful quantity for short-range problems.

Formula for hitting space at very short ranges

space S' is defined approximately by the relation

$$S = h \cot \omega \quad (1701)$$

in which ω is the angle of fall corresponding to the target distance X .

Although no rigorous proof of equality between the angles ω and λ (as illustrated in Figure 51) is available, it has been established by exhaustive comparative solutions that these angles are indeed very nearly equal in any practical situation. That is to say, solutions for the hitting space according to the approximate formula $S' = h \cot \omega$ and according to more accurate formulas or processes,* have been compared for a great variety of assumed ranges and target heights, with the result that even at the shortest ranges and for the greatest target heights that are likely to occur in practice, the maximum error of the approximate formula has been found to be about 1%.

1704. We have found in article 921 that Column 19 is computed from the formula

$$\Delta h = \Delta X \tan \omega \quad (907)$$

whence

$$\Delta X = \Delta h \cot \omega.$$

Thus Column 19 gives the height of target which corresponds to a hitting space of 100 yards, and if we divide the given height of target by the value from Column

Limitations of Column 19 for determining hitting space at very short ranges

19 and multiply the result by 100, we will have the hitting space for the given target. But the usefulness of Column 19 in this connection is limited by the fact that the values in this column are tabulated only to the nearest whole foot and hence may be in error by as much as $(\pm) 1/2$ foot. For example, in the 4''2900 f.s. range table we find that the value in Column 19 at 1700 yards is 3 feet. Since

this may actually be 3.49 feet, the error resulting from the use of the value as

tabulated may be as great as $\frac{.49}{3}$, or about 16%. The following example shows

how great an error can arise from this cause. At the range 1700 yards for the 4'' 2900 f.s. gun the angle of fall equals $0^\circ 44'$; according to formula (1701), the correct hitting space for a 20-foot target for this gun at 1700 yards is then

$h = 20$ feet.....	$\log 1.30103$
$\omega = 0^\circ 44'$	$\operatorname{cosec} 1.89280$
$S' = 1563$ feet.....	$\log 3.19383$
$= 521$ yards	

But by using Column 19 we get

$$\frac{20}{3} \times 100 = \underline{667 \text{ yards.}}$$

According to the information obtained by means of Column 19 the spotter for a destroyer at Short Range Practice might then expect that a shot which landed 667 yards behind the 20-foot target should have passed through the top edge

* An accurate solution for the hitting space can be made by solving the trajectory for the abscissa x which corresponds to the ordinate $y = h$. This can be accomplished either by numerical integration or by means of Ingalls' formula (1629).

of the target, whereas, in fact, such a shot would actually have passed well over the top of the target and have landed 146 yards beyond the limit of the hitting space of the 20-foot target. The unsuitability of Column 19 for the determination of the hitting space for short ranges is further illustrated by the following example. In the 4''2900 f.s. range table the value 3 feet appears in Column 19 for ranges from 1300 yards to 1700 yards; hence by use of this column we would get the same hitting space for any range from 1300 yards to 1700 yards. According to formula (1701) we find, however, that the hitting space for a 20-foot target actually varies from 716 yards at the range 1300 yards, to 521 yards at the range 1700 yards. The above examples make it clear that the hitting space for short ranges should always be determined by means of formula (1701).

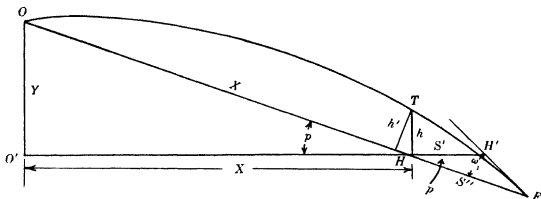


FIGURE 52

Effect of angle of position on hitting space 1705. At very short ranges the hitting space is affected materially by the angle of position, making it necessary to perform further operations on the value of hitting space obtained from formula (1701).

In Figure 52 the position of the gun is at O , at a height Y feet above the surface of the water, and the target is resting on the surface of the water at H ;

the angle of position is $p = \tan^{-1} \frac{Y}{X}$. In any practical case the angle of position is

so small that no distinction need be made between the inclined distance OH and the horizontal distance $O'H$; for example, with the gun as high as 60 feet and the range as short as 1600 yards, the difference between OH and $O'H$ is only about 0.1 yard. Hence we may consider OH to be equal to the horizontal target distance X . Also, the difference between the vertical height h of the target and its perpendicular height h' on the line of position, is negligible; with the gun as high as 60 feet and the range as short as 1600 yards, the difference between h and h' , in the case of a 20-foot target, is only about .001 foot.

The hitting space in Figure 52 is $S' = HH'$, H' being the point where the trajectory $OTH'F$ intersects the plane of the water. But if ω is the angle of fall for the range $X = OH$, then the formula $h \cot \omega$ gives, in this case, not the actual hitting space $S' = HH'$, but an imaginary hitting space $S'' = HF$ measured along the line of position OHF . We shall call this imaginary hitting space S'' the *inclined hitting space*. In order to determine the actual hitting space, S' , which is measured on the plane of the water, we proceed as follows. Having found S'' from the relation $S'' = h \cot \omega$, we add S'' to X , thus obtaining the range $X + S'' = OF$.

With this new range we find the angle of fall, ω_1 , from the range table. Considering the arc $H'F$ to be a straight line, which may be done without material error in any practical case, we have the triangle $HH'F$ in which the side $S'' = HF$ and the two angles, p and ω_1 , are known. The triangle can now be solved for the side $S' = HH'$ by means of the trigonometric formula

$$S' = S'' \sin \omega_1 \operatorname{cosec} (\omega_1 + p). \quad (1702)$$

The procedure for finding the hitting space S' of a target of height h at a horizontal range X , when the gun is elevated Y feet above the water and the target is resting on the water, is then as follows.

$$(a) \quad S'' = h \cot \omega, \quad \tan p = \frac{Y}{X}$$

Determination of hitting space when gun is above target and target rests on the water

in which S'' is the inclined hitting space and ω is the angle of fall at the given range X to the target.

$$(b) \quad X_1 = X + S''$$

and with X_1 the corresponding angle of fall, ω_1 , is found from the range table.

$$(c) \quad S' = S'' \sin \omega_1 \operatorname{cosec} (\omega_1 + p).$$

1706. At Short Range Practice the target screen is spread so that its bottom edge is from four to six feet above the water, and in this case the hitting space is not bounded by the foot of the target. This situation is illustrated in Figure 53. The bottom edge of the target is now d feet above the water; other features are the same as in Figure 52. The hitting space S' of the target h is, in this case, evidently equal to the difference between the hitting spaces for target heights equal to d and $(h+d)$, respectively. If the hitting space for the target height d is denoted by S'_1 and the hitting space for the target height $(h+d)$ is denoted by S'_2 , the hitting space for the target h is then equal to $S' = S'_2 - S'_1$. The procedure for finding S'_1 and S'_2 is similar to that outlined in the preceding article.

The complete procedure for finding the hitting space S' when both the gun and the target are elevated above the surface of the water, is outlined under Figure 53; the meanings of the symbols used in this outline are clearly illustrated in Figure 53.

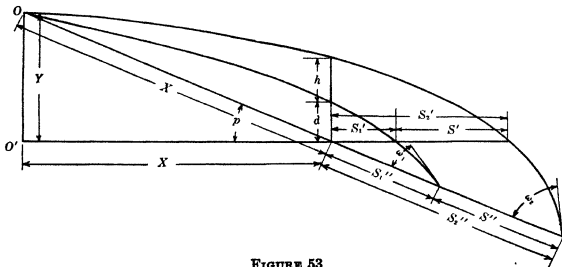


FIGURE 53

Hitting space
for a gun
elevated
above the
target, when
the target is
above the
water

$$(a) \quad S''_1 = d \cot \omega, \quad \tan p = \frac{Y}{X}$$

$$S''_2 = (h+d) \cot \omega$$

ω being for the range X to the target.

$$(b) \quad \text{With range } X+S''_1, \text{ get } \omega_1 \text{ from the range table.}$$

$$\text{With range } X+S''_2, \text{ get } \omega_2 \text{ from the range table.}$$

$$(c) \quad S'_1 = S''_1 \sin \omega_1 \operatorname{cosec} (\omega_1 + p)$$

$$S'_2 = S''_2 \sin \omega_2 \operatorname{cosec} (\omega_2 + p)$$

$$(d) \quad S' = S'_2 - S'_1$$

1707. The following example illustrates the application of the process outlined above and also shows how greatly the angle of position affects the hitting space at short ranges.

Given: A 4''2900 f.s. gun is to fire at Short Range Practice; the mean range will be 1633 yards. The gun is elevated 25 feet above the water. The target screen is 15 feet high and is spread so that its bottom edge is 5 feet above the water.

Find: The hitting space for the 15-foot target at the mean range of 1633 yards. What error would result from using Column 19 and neglecting the angle of position? (Use angles to the nearest whole minute.)

Following the procedure outlined in article 1706, we have

$$(a) \quad \tan p = \frac{Y}{X}$$

$Y = 25$ feet.....	log 1.39794
$X = 4900$ feet.....	log 3.69020.....
$p = 0^\circ 18'$	ltan 7.70774-10

$$S''_1 = d \cot \omega$$

$$S''_2 = (h+d) \cot \omega$$

From the range table, at the range 1633 yards, we find $\omega = 0^\circ 42'$.

$d = 5$ feet.....	log 0.69897	$(h+d) = 20$ feet.....	log 1.30103
$\omega = 0^\circ 42'$	lcot 1.91300	$\omega = 0^\circ 42'$	lcot 1.91300
$S''_1 = 409.2$ feet.....	log 2.61197	$S''_2 = 1636.9$ feet.....	log 3.21403

$$(b) \quad X + S''_1 = 5309 \text{ feet or } \underline{1770 \text{ yards}}$$

whence, from the range table, $\omega_1 = 0^\circ 46'$.

$$X + S''_2 = 6537 \text{ feet or } \underline{2179 \text{ yards}},$$

whence, from the range table, $\omega_2 = \underline{1^\circ 00'}$.

$$(c) \quad S'_1 = S''_1 \sin \omega_1 \operatorname{cosec} (\omega_1 + p)$$

$$S'_2 = S''_2 \sin \omega_2 \operatorname{cosec} (\omega_2 + p)$$

$S''_1 = 409.2$ feet... log	2.61197	$S''_2 = 1636.9$ feet... log	3.21403
$\omega_1 = 0^\circ 46'$	lsin 8.12647-10	$\omega_2 = 1^\circ 00'$	lsin 8.24186-10
$(\omega_1 + p) = 1^\circ 04'$	lcsc 1.73012	$(\omega_2 + p) = 1^\circ 18'$	lcsc 1.64422
$S'_1 = 294$ feet... log	2.46856	$S'_2 = 1259$ feet..... log	3.10011
$= \underline{98 \text{ yards}}$		$= \underline{420 \text{ yards}}$	

$$(d) \quad S' = S'_2 - S'_1 \\ S' = 420 - 98 = \underline{322 \text{ yards}}$$

Had we used Column 19 and neglected the angle of position, we would have found

$$S'_1 = \frac{5}{3} \times 100 = 167 \text{ yards,}$$

$$S'_2 = \frac{20}{3} \times 100 = 667 \text{ yards,}$$

$$S' = 667 - 167 = \underline{500 \text{ yards,}}$$

and our result would have been in error 178 yards, or about 55%.

Had we used the hitting-space formula (1701) but neglected the angle of position, our result would have been,

$$S''_2 - S''_1 = 1637 - 409 = 1228 \text{ feet or } \underline{409 \text{ yards}}$$

and this result would have been in error 87 yards, or about 27%.

1708. The above example also illustrates the correct method for determining the relation between points of impact in the vertical plane of the target and on the surface of the water, at very short ranges. The spotter would know, in the case illustrated, that a shot through the top edge of the target should fall at the distance $S'_2 = 420$ yards beyond the target, and a shot through the bottom edge at the distance $S'_1 = 98$ yards beyond the target. The distance beyond the target at which a shot through the bull's-eye should fall can be found by the same process. The spotter uses this information for constructing a *splash diagram*.

1709. A composite graph of hitting-space curves provides a convenient means for finding the data required for the construction of a splash diagram. Such a *hitting-space diagram* may be prepared for each battery carried on a ship, for the mean range of Short Range Practice, and for all heights of gun and target that are likely to occur. The following is an outline of a method by means of which a hitting-space diagram may be constructed.

- (a) Assume a range which will be the mean range for the given battery at Short Range Practice.
- (b) Assume zero height of gun, and compute the hitting space for target heights 0, 3, 5, 10, 15, etc. feet, up to the greatest target height desired. Plot the values of hitting space against height of target, and mark the graph to indicate that it applies to zero height of gun. This graph should be a straight line, i.e., the graph of $h \cot \omega$, ω being the angle of fall corresponding to the assumed mean range.
- (c) Assume heights of gun of 5, 10, 15, 20, etc. feet, up to the greatest height of gun likely to occur. For each of these heights of gun repeat the process outlined in (b), and mark the resulting graphs according to the heights of gun to which they apply. For other than zero height of gun these graphs are curves.

Connect the points of the several height-of-gun curves which correspond to the same target heights, thus obtaining a series of height-of-target curves that intersect the height-of-gun curves.

A hitting-space diagram for the 4"2900 f.s. gun at the range 1633 yards is illustrated in Plate V. The problem stated in article 1707 can be solved graphically by means of this diagram, as follows. The gun is 25 feet above the water, and the top of the target 20 feet above the water. On the diagram (Plate V), the 25-foot height-of-gun curve and 20-foot height-of-target curve intersect at *A*; by projecting this point vertically down to the hitting-space scale we find $S'_2 = 420$ yards. The bottom of the target is 5 feet above the water. The 25-foot height-of-gun curve and 5-foot height-of-target curve intersect at *B*, whence we find $S'_1 = 98$ yards. The hitting space is therefore $420 - 98 = 322$ yards, which agrees with the result found by computation. The bull's-eye of the target is 12.5 feet above the water. The 25-foot height-of-gun curve and 12.5-foot height-of-target curve intersect at *C*, whence we find that a shot passing through the bull's-eye will fall 253 yards beyond the target.

Using the zero height-of-gun curve with the above data we find

$$S'_2 = 546 \text{ yards, } S'_1 = 137 \text{ yards}$$

$$S' = 546 - 137 = 409 \text{ yards}$$

which agrees with the result found in article 1707 when height of gun was neglected.

EXERCISES

1. *Given:* The type of gun, the range, the height of target, depth of target in the line of fire, height of gun above the water, and height of bottom of target above the water.

Find: The hitting space, using the range table as required.

	Given						Answers
	Gun	Range (yds.)	Target (feet)			Height of gun (feet)	Hitting Space (yds.)
			Height	*Depth	Bottom above water		
A	4"2900 f.s.	2500	14	0	4	26	193
B	6"2600 f.s.	1800	15	0	6	28	293
C	12"2900 f.s.	1650	15	0	5	35	342
D	12"2900 f.s.	20,000	30	90	0	..	62

* See art. 1402 (h).

APPENDIX C

THE SOLUTION OF INCLINED TRAJECTORIES. GENERAL FEATURES INVOLVED IN THE CONTROL OF ANTI-AIRCRAFT FIRE AND ILLUMINATION FIRE.

Introductory Note

The purpose of this appendix is to explain the methods by which range tables, or their equivalent, are prepared for anti-aircraft fire and illumination fire, and to set forth the theoretical bases on which the control of these types of fire rests. In order that this purpose may be accomplished it is desirable to discuss some practical features of the control problems. Such discussions should, however, be regarded as fundamental in their nature, and not as being designed to acquaint the student with practical details. Moreover, this text is limited to comparatively general references to practical methods actually in use for the control of anti-aircraft fire, since many features of such methods are regarded as confidential. The anti-aircraft tracking sheet illustrated in this appendix is obsolete, but it is useful, nevertheless, for demonstrating the principles that enter into the construction and use of tracking sheets of more recent issue. It is expected that tracking sheets and other graphical devices in current use will be made available for supplemental classroom study.

1801. The problem of computing anti-aircraft trajectories is unique principally in that it involves large angles of position. Numerous approximate methods have been devised for dealing with greatly inclined trajectories, most of them based, in one form or another, upon the principle of tilting the horizontal trajectory.* These methods have generally been discarded in favor of the direct and relatively accurate method based on numerical integration; the principal features of the latter method are outlined below.

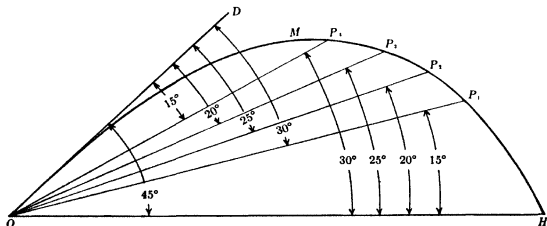


FIGURE 54

1802. In Figure 54, OMH represents a horizontal trajectory with angle of departure 45° and horizontal range $X = OH$. Let us suppose that this trajectory

* Ref. Chapter VIII, *Handbook of Ballistics*, Vol. I, Cranz and Becker.

has been computed by the method of numerical integration and plotted to large scale. (The trajectory OMH in Figure 54 represents, approximately to scale, the 45° trajectory of the 5"3150 f.s. gun, the horizontal range being equal to about 23,000 yards and the maximum ordinate to about 24,000 feet.) The lines of position OP_1, OP_2, OP_3, OP_4 , make with the horizontal the angles of position, respectively, 15°, 20°, 25°, and 30°. Each of these lines of position cuts off a segment of the horizontal trajectory, and each segment is an inclined trajectory. For example, the line of position OP_1 cuts off the inclined trajectory OMP_1 ; the angle of position of the latter trajectory is equal to 15°, and the angle of elevation ϕ' is evidently the angle DOP_1 , which is equal to 45° - 15° = 30°. The slant range, $X' = OP_1$, can be measured graphically, or it can be determined from the relation $X' = x \sec p$, x being the abscissa of the point P_1 . The time of flight (t), angle of inclination (θ), and remaining velocity (v) corresponding to the point P_1 on the horizontal trajectory OMH being known, we have the terminal elements of the inclined trajectory OMP_1 .

By the same process we can determine the terminal elements of a number of inclined trajectories, each corresponding to a different angle of position. All of these trajectories are, of course, portions of the same 45°, horizontal trajectory, but, considered as inclined trajectories, each of them has a different angle of elevation. From the one horizontal trajectory and several lines of position plotted in Figure 54, we can derive the following inclined trajectories:

- For $p = 15^\circ$ and $\phi' = 30^\circ$, $X' = OP_1$
- For $p = 20^\circ$ and $\phi' = 25^\circ$, $X' = OP_2$
- For $p = 25^\circ$ and $\phi' = 20^\circ$, $X' = OP_3$
- For $p = 30^\circ$ and $\phi' = 15^\circ$, $X' = OP_4$

These results are illustrated in the figure. By plotting at suitable intervals along the horizontal trajectory OMH the values of t , θ , and v found in the step-by-step solution of that trajectory by numerical integration, the terminal values of these elements for any inclined trajectory whose terminal point lies on the given horizontal trajectory can be determined very conveniently.

1803. A single horizontal trajectory yields, of course, only one inclined trajectory corresponding to any given angle of position, i.e., only one value of ϕ' corresponding to each assumed value of p . Additional inclined trajectories, for

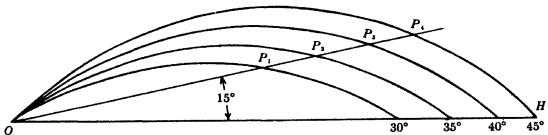


FIGURE 55

any assumed angle of position, can be obtained by plotting additional horizontal trajectories, as shown in Figure 55. This figure represents a plot (not to scale) of horizontal trajectories having angles of departure 30°, 35°, 40°, and 45°, and of a 15° line of position. From this plot we can evidently derive the following inclined trajectories:

For $p = 15^\circ$ and $\phi' = 15^\circ$, $X' = OP_1$

For $p = 15^\circ$ and $\phi' = 20^\circ$, $X' = OP_2$

For $p = 15^\circ$ and $\phi' = 25^\circ$, $X' = OP_3$

For $p = 15^\circ$ and $\phi' = 30^\circ$, $X' = OP_4$

The other terminal elements for each of these trajectories can be found as already described in the foregoing article.

1804. We have already seen (art. 819) how a range table for horizontal fire is built up by a process of interpolation between computed points. A range table for inclined fire could be built up by a similar process, but such a table would have to include angle of position as an additional argument and would be too complicated for practical use. A more practical arrangement of data pertaining to anti-aircraft trajectories is obtained as follows.

Horizontal trajectories are computed and plotted for angles of departure up to 90° , at intervals of 5° . Lines of position are plotted for angles of position up to 90° , at intervals of 10° . The process described in article 1803 and il-

Graphs of
data for
anti-aircraft
trajectories

lustrated in Figure 55 is carried out for each plotted line of position. This yields, for each angle of position, a series of corresponding slant ranges and angles of elevation, all pertaining to that angle of position.

For example, the point where the 10° line of position cuts the 15° trajectory gives the slant range corresponding to the angle of elevation 5° ; the point where the 10° line of position cuts the 20° trajectory gives the slant range corresponding to the angle of elevation 10° ; etc. For each angle of position, the various slant ranges thus obtained are plotted against their corresponding angles of elevation, and a fair curve is drawn through the plotted points. This process yields a series of graphs, each graph representing the range-elevation relation for one angle of position. The angle of elevation corresponding to *any* slant range and any *plotted* value of angle of position can then be taken directly from one of the graphs, and the angle of elevation corresponding to *any* slant range and *any* angle of position can be found by interpolating between two graphs. Graphs showing the relation of time of flight to slant range and angle of position are prepared similarly.

1805. The effects of wind, and of variations from standard in initial velocity, weight of projectile, and atmospheric density are determined by the method of differential variations (see note on p. 124). As any inclined trajectory may be considered to be a portion of a horizontal trajectory (as illustrated in Figure 54), the determination of these variations for an inclined trajectory can be simplified by the use of weighting factors based on the horizontal trajectory which includes the given inclined trajectory. Graphs are prepared to show the effects of variations from standard conditions on angle of elevation and time of flight.

1806. The burning rate of powder-train time fuzes varies with atmospheric density and pressure, and hence with altitude; consequently the fuze settings for this type of fuze do not necessarily agree with actual times of flight. The relation

Relation be-
tween time
of flight
and fuze
setting

between fuze setting and actual time of flight is determined experimentally for each type of powder-train time fuze; graphs are prepared to show directly the relation of fuze setting to slant range and angle of position, under standard atmospheric conditions. Additional graphs are prepared for the determination of corrections to

these fuze settings for variations from standard in atmospheric density and pressure. The burning rate of the powder train depends also on the velocity of the projectile. A variation in the initial velocity therefore requires a change in fuze

setting due to its effect on the burning rate of the fuze, as well as a change due to its effect on the time of flight.

1807. The drift formulas given in Chapter 9 (art. 911 and note adjacent thereto) are of such nature that they are applicable only to the terminal point of a horizontal trajectory; that is to say, the drift for an inclined trajectory may not be found from these formulas by substituting in them the values of elements for the intermediate point of a horizontal trajectory which corresponds to the terminal point of the given inclined trajectory. The drift for intermediate points of a horizontal trajectory can be determined satisfactorily only by step-by-step integration of analytical expressions for this element. It is necessary, of course, to determine constants for use in these expressions by experimental firing.* Having thus determined the drift for various points along a horizontal trajectory, the drift for inclined trajectories whose terminal points lie on this horizontal trajectory can be determined as already described for other elements (art. 1802).

A very rough approximation of the drift for an inclined trajectory is given by the relation

$$D_{\text{inclined}} = D_{\text{surface}} \times \cos p$$

in which D_{inclined} is the drift for an inclined trajectory, D_{surface} is the drift for a horizontal trajectory having an angle of elevation equal to that of the inclined trajectory, and p is the angle of position of the inclined trajectory. This is a purely arbitrary relation, based on the consideration that when $p=90^\circ$ the fire is vertical and there is no drift, for under this condition there is no curvature to the trajectory and consequently no moment causing precession. It is assumed, then, that since the drift is reduced to zero at the zenith, it is related to the angle of position by a function which is equal to unity for zero degrees and to zero for ninety degrees. $\cos p$ is used merely because it satisfies these conditions. Although the approximate relation given above was formerly used for determining the drift of inclined trajectories, it is the present practice to solve for this element by the method outlined in the foregoing paragraph.

1808. A sight scale graduated in terms of range evidently is unsuitable for antiaircraft fire, for in this type of fire the angle of elevation is governed by the angle of position as well as by the range. Sight scales for antiaircraft fire are accordingly graduated in minutes of arc, and are set directly in terms of angle of elevation, or *sight angle*, rather than in terms of range. (The *sight angle* is the angle, measured in the plane of elevation, between the line of sight of a gun and the axis of the bore of the gun.)

The various graphs described above constitute the equivalent of a range table for an antiaircraft gun, and by means of them the angle of elevation required for a given slant range and angle of position, under any specified conditions, can be determined by methods similar to those followed in determining the sight-bar range for surface fire from a range table. The graph of angle of elevation vs. slant range and angle of position, gives the angle of elevation corresponding to any specified slant range and angle of position under standard conditions. Other

* Neither of the formulas given in Chapter 9 is in shape for this process. Analytical expressions in shape for solution by numerical integration are given on p. 353, *Handbook of Ballistics*, Vol. I, Cranz and Becker, and on pp. 119-120, *Computation of Firing Tables for the U. S. Army*, H. P. Hitchcock.

graphs give the corrections to the angle of elevation required for variations from standard in initial velocity, atmospheric density, and for wind, etc. The deflection for antiaircraft fire can also be determined by methods similar to those already described for surface fire, the essential difference being that the required data are found from graphs instead of from the columns of a range table.

1809. It is obviously impracticable to use the graphs directly, in the manner outlined above, for continuous determination of the angle of elevation, deflection, and fuze setting under the condition of a rapidly changing slant range and angle of position. The only wholly satisfactory solution of the problem of determining these elements with the required rapidity lies in a mechanical system designed to generate them continuously. Such systems are now installed on most vessels.

Mechanical systems for control of anti-aircraft fire

The general principles of methods which were developed prior to the adoption of mechanical systems are described below. These methods still serve as the primary methods of antiaircraft fire control for vessels not equipped with mechanical systems, and as stand-by methods for vessels having the mechanical equipment.

Manual systems for control of anti-aircraft fire

1810. Plate VI is a reproduction of an *antiaircraft tracking sheet* for a 3" antiaircraft gun. The construction of this diagram is similar to that of Figure 55. On it are plotted horizontal trajectories for angles of departure from 5° to 85° at intervals of 1°. The trajectories corresponding to whole multiples of 5° are labeled where they meet the inner margin of the diagram, and along each of these trajectories are marked times of flight at intervals of $\frac{1}{10}$ second, labeled at each $\frac{1}{2}$ second. A scale for measuring angles of position is laid off on the outer margin. Circles concentric with the origin (lower left-hand corner) indicate the slant range to any point on any trajectory; these range circles are labeled along the bottom edge of the diagram. Horizontal lines across the diagram indicate altitude; they are labeled along the left-hand edge of the diagram.

Antiaircraft tracking sheet

1811. The antiaircraft tracking sheet serves two purposes; it is a convenient tracking sheet for plotting the approach of the aircraft, and it provides a means for deducing directly from the plotted track the angle of elevation and fuze setting required to hit the aircraft at any point in this track.

In order to plot the track of the approaching aircraft it is necessary to measure its angle of position and slant range (or altitude) at several points, as, for example, at P_1 , P_2 , and P_3 . Plate VI shows the track of a plane approaching at an altitude of 5000 feet. By measuring the distances covered by the plane between successive positions along its track, and comparing these with the times elapsed between these positions, an estimate of the plane's speed may be made; the speed thus found is, of course, the plane's relative speed with respect to the firing ship.

Now let us suppose that the plane is to be fired at when it reaches point P_4 . The line of position, which is the line drawn from the origin through P_4 , intersects the marginal scale at 27°, indicating that at point P_4 the angle of position of the plane will be 27° ($p = 27^\circ$). The point P_4 also lies on the 30° trajectory (as indicated at the inner right-hand margin). Therefore the angle of elevation required to hit the plane at P_4 is $\phi' = \phi - p = 30^\circ - 27^\circ = 3^\circ$. Also, at P_4 the time of flight is 7.5 seconds.

But if we should wait until the plane actually reaches the point P_4 before opening fire, the plane would, during the time of flight of the projectile, reach a point considerably beyond P_4 ; for example, if the plane were traveling at 90 knots

Determination of
angle of eleva-
tion from the
antiaircraft
tracking sheet

it would cover 375 yards during the time of flight of 7.5 seconds. Putting the plane 375 yards back along its track locates it at point D , where its angle of position is $24\frac{1}{2}^\circ$. Then in order to hit the plane at P_4 we must fire upon it when it reaches point D ; since P_4 still lies on the 30° trajectory, while D lies on the $24\frac{1}{2}^\circ$

line of position, the required angle of elevation is $30^\circ - 24\frac{1}{2}^\circ = 5\frac{1}{2}^\circ$, or $330'$. What we have done is to find the trajectory that will *intercept* the target, rather than the one that will pass through the position occupied by the target at the instant of fire. The problem is, of course, similar to that of firing at a moving target on the surface. If the sight angle $330'$ is set on the sight scale and the gun is aimed at the target when the latter occupies point D , the bore will be elevated as required to cause the projectile to follow the trajectory which will intercept the target at point P_4 .

The times plotted along the trajectories of Plate VI are actual times of flight and hence can be used directly as fuze settings only for mechanical time fuzes. Fuze settings applicable to a particular type of powder-train time fuze can also be plotted along the trajectories. For the purposes of the illustrations to be given here we shall assume that the times plotted on Plate VI represent fuze settings. The fuze setting required to cause a burst at point P_4 is then 7.5 seconds.

1812. The great rapidity with which the problem must be solved renders direct use of the diagram, as outlined above, impracticable. An understanding of the direct, graphical solution on the antiaircraft tracking sheet, as outlined above, is essential, however, to a proper understanding of the methods actually employed. These methods are based on the use of tabulated solutions for the sight angle and fuze setting corresponding to various combinations of target speed and target altitude.

Referring to Plate VI, it will be noted that after the track of the target has been established, altitude and angle of position define the target's position at any point in this track. Thus if the target is flying at an altitude of 5000 feet and its angle of position is 27° , it must occupy the point P_4 . For tabulated solutions, altitude is a more convenient argument than slant range, for the target's altitude often is nearly constant whereas its slant range always changes rapidly.

A table of sight angles and fuze settings corresponding to a single altitude and various target speeds can be prepared as follows. For this illustration we shall assume an altitude of 4000 feet and a target speed of 90 knots (90 knots corresponds to 50 yards per second). Now let us assume for target positions a series of points defined by certain fuze settings, say 12, 10, and 8 seconds. On Plate VI these positions are plotted on the 4000-foot altitude line and are marked 12, 10, and 8.

In order to hit the target at these positions, we must fire on it before it reaches them. In order to hit the target at the 12-second point we must fire at it when it is still $12 \times 50 = 600$ yards from this point, or at the position marked Δ and labeled Z-12. In order to be hit at the 10-second and 8-second points the target must be fired at when it occupies the positions Z-10 and Z-8, respectively; Z-10 is located $10 \times 50 = 500$ yards from the 10-second point, and Z-8 is located $8 \times 50 = 400$ yards from the 8-second point.*

* The times to be used for determining these distances should, of course, be the *times of flight* corresponding to the given fuze settings; in accordance with the assumption stated in article 1811 (par. 5), times of flight are here taken to be equal to fuze settings.

The sight angles required to hit the target at the 12-, 10-, and 8-second points can now be found by the process described in article 1811. The 12-second point lies on the $21^{\circ}40'$ trajectory, and the angle of position at Z-12 is equal to $13^{\circ}50'$; hence the sight angle required for the 12-second point is equal to $21^{\circ}40' - 13^{\circ}50' = 7^{\circ}50'$, or $470'$. By the same process the sight angle for the 10-second point is found to be equal to $22^{\circ}10' - 15^{\circ}40' = 6^{\circ}30'$, or $390'$, and for the 8-second point $23^{\circ}45' - 18^{\circ}25' = 5^{\circ}20'$, or $320'$.

1813. All of the above solutions pertain to a 90-knot target flying at an altitude of 4000 feet. Now let us refer to the positions for which we have made solutions, as *zones*; that is, let us call the positions Z-12, Z-10, and Z-8, respectively, Zone 12, Zone 10, and Zone 8. These zones have been so chosen and designated that the zone description of a given zone agrees with the fuze setting required to hit a target which is fired upon when it occupies that zone; the sight angles for the several zones* have been found as shown above. The information thus found may now be arranged in tabular form as follows:

		Altitude 4000 feet									
Antiaircraft sight-setting table	Zone	2	3	4	5	6	7	8	9	10	12
	Target Speed										
	70										
	80										
	90							320		390	470
	100										
	110										

The above table, which is called a *sight-setting table*, represents a convenient form for tabulating all of the solutions pertaining to the 4000-foot altitude. The three solutions actually made in the foregoing article have been entered in the table. In order to complete the table, solutions must be made for each zone and each target speed included in the arguments of the table. It must be borne in mind that all of the values included in a single table pertain to a single altitude. A complete outfit of sight-setting tables includes a separate table for each 1000 feet of altitude up to the greatest altitude likely to be required.

1814. The sight-setting table is used as follows. As soon as the altitude of the target has been determined, the appropriate sight-setting table is defined. As soon as the speed of the target has been determined, the appropriate line of the sight-setting table is defined, and the sight angles corresponding to the various zones through which the target has to pass are then immediately available. For example, let us suppose that the altitude and speed of an approaching bomber have been determined, by tracking, to be 4000 feet and 90 knots, and that it has been decided that the position Z-12 (Plate VI) is the earliest at which it is practicable to open fire. From the sight-setting table for the altitude 4000 feet it is found that the sight angle corresponding to Zone 12 and target speed 90 knots, is $470'$; fuzes are then set at 12 seconds and sight scales at $470'$, and fire is opened as the target approaches the position Z-12. After the target has passed beyond Z-12, the fuze setting and sight angle are reset for a new zone, say Zone 10, and the fire is continued.

In determining the proper instants at which to open fire for the zone initially selected, and at which to change the settings for a new zone, due allowance must

* The reason for referring to these points as *zones* is explained in article 1814.

Allowance for dead time be made for *dead time*. This can be done readily by advancing the actually plotted target positions a distance equal to the product of the dead time and target speed. Foreexample, if the dead time is equal to 10 seconds and the target speed is equal to 90 knots, the target is considered to have reached a given position when, according to the plotted track, it is still $10 \times 90 = 900$ yards from that position.

Allowance must be made also for the length of the interval during which the settings are to remain unchanged. Thus if the settings are to be changed at intervals corresponding to a change of one second in fuze setting, fire for a given zone should be opened, with the settings for that zone, when the target reaches a position about half-way between the given zone and the next preceding zone, and continued until the target reaches a position about half-way between the given zone and the next following zone (allowance for dead time being made, of course, as indicated above). The size of interval to be used depends on the rate of fire of the battery and on the rapidity with which the required settings can be found, transmitted, and applied. Intervals corresponding to a change of only one second in fuze setting will probably be much too short when the target speed is great; even in the case illustrated on Plate VI, the intervals corresponding to changes of one second in fuze setting are equal to only about seven seconds of time. It appears, then, that it is impracticable to assign definite limits to zones; Zone 10, for example, may extend from the point corresponding to the fuze setting 10.5 seconds to the point corresponding to the fuze setting 9.5 seconds, or it may extend from the 11-second point to the 9-second point, or from the 11.5-second point to the 8.5-second point, etc., depending upon the interval chosen by the control party. It is for this reason that the term *zone* is applied to a *point*, such as the points Z-8, Z-10, and Z-12 (Plate VI). Actually these points represent the middle points of their respective zones, and the width of the zones depends upon the intervals to be used.

Width of zones 1815. It is to be noted that the points Z-8, Z-10, and Z-12, as plotted on Plate VI, define the positions of Zone 8, Zone 10, and Zone 12, respectively, for the single target speed of 90 knots, as well as for the single target altitude of 4000 feet. In order that the diagram may be used in the manner outlined in the foregoing article, it is necessary to plot on it zone positions for all target speeds and altitudes that are likely to occur in practice.

Provisions for handling various altitudes and target speeds on the tracking sheet

The method for locating the position of a given zone for a given target speed and altitude has already been explained in article 1812. Let us suppose that the positions for Zone 8, for the target speed 90 knots, have been located on the 1000-foot, 2000-foot, 3000-foot, etc. altitude lines, just as the point Z-8 has been located on the 4000-foot altitude line. A curve connecting these positions will be the locus of Zone 8 for the target speed 90 knots and *any altitude* (within the limits of the diagram). The loci of other zones can be plotted similarly. The complete diagram will then include a group of curves for each zone, each group containing curves which are the loci of positions corresponding to the same fuze setting (or zone description) and to various target speeds.*

1816. It is to be understood, of course, that the diagram described above, and its accompanying sight-setting tables, are based on standard conditions, i.e.,

* It is expected that the diagrams actually in use in the service will be exhibited in the classroom in connection with the study of this appendix. Reproduction of these diagrams in this textbook is not permitted.

range-table conditions, with the exception that target speed has been taken into account in plotting the various zone loci on the diagram and in determining the corresponding sight angles for the sight-setting tables. Other ballistic corrections can be found from the graphs described in articles 1806-1808, but in view of the form in which the firing data have been prepared it is very inconvenient to apply corrections to them in the usual manner. One way of handling such corrections is to express them in terms of time of flight and to apply them, in this form, to the zone description. This can be done (in advance of firing, of course,) by comparing the sight-angle and fuze-setting corrections corresponding to variations from standard initial velocity and atmospheric density, with the data of the sight-setting tables for the conditions of altitude and target speed likely to be encountered, and thus deducing appropriate corrections in terms of zone description.

Since the target is tracked with reference to a fixed origin on the diagram (point *O*, Plate VI), the target speed found is the target's *relative* speed with respect to the firing ship; that is, the speed found is the resultant of target speed and firing-ship speed. This means that the firing ship's speed is accounted for in the target speed, and consequently it is accounted for as target speed. According to the rules stated in article 1219, corrections for wind should then be based on the *apparent wind*. (This applies, of course, only to the stand-by methods; antiaircraft range-keepers handle target speed and firing-ship speed separately, and the considerations governing the choice of true or apparent wind for use with an antiaircraft range-keeper are the same as for any other range-keeper (art. 1220).)

1817. In a bombing attack, the motion of the attacking bomber must be such as to cause the component of his own motion across the line of fire to be practically the same in amount, and in the same direction, as the component of the firing ship's motion across the line of fire. If the correction for wind across the line of fire is based on the *apparent* wind, the corrections for target motion and firing-ship motion across the line of fire then practically cancel each other, and the deflection need include corrections only for drift and apparent wind. The deflection problem in a bombing attack is therefore relatively simple. In a torpedo attack, the motion of the attacking torpedo-plane may be such as to have a large and rapidly varying component across the line of fire, and the deflection problem in this type of attack is therefore much more difficult.

**Deflection
for antiair-
craft fire**

CONTROL OF ILLUMINATION FIRE.

1818. The object of illumination fire is to release a source of light above and behind a target in order that the latter may be silhouetted against an illuminated water area and thus rendered visible to the firing vessel at night. An illuminating projectile, or starshell, contains an illuminating element, or star, which is attached to a parachute, and an expelling charge which, when set off by a time fuze, ignites the star and releases it and its parachute. The duration of burning of the star and its rate of descent are such that the star should be released at an altitude of about 1000 feet in the case of 3" starshell, and at an altitude of about 1500 feet in the case of 4", 5", and 6" starshell. It has been found by experience that the point of release should be sufficiently far behind the target to prevent the illumination of water in front of the

**General
principles
of illumina-
tion fire**

target; the purpose of the star is to silhouette the target, and illumination in front of the target reduces the silhouette effect. Range tables for illumination fire are based on a release point 1000 yards beyond the target. The general principles involved in the preparation of these range tables are outlined below.

1819. In Figure 56, a gun is located at O and a target at T . The trajectory of the starshell must be such that it will have an ordinate $P'P$, equal to about 1500 feet (assuming a 5" gun), where its abscissa is OP' , equal to OT plus 1000 yards. Let us suppose, now, that a trajectory has been plotted for the angle of departure ϕ , giving the horizontal range OH . We may determine graphically at what point, P , this trajectory has an ordinate of 1500 feet, and also find the abscissa OP' of

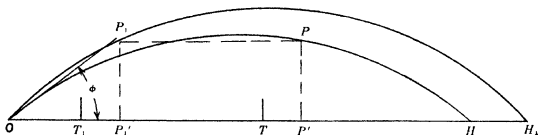


FIGURE 56

this point. This trajectory will then be the proper one for a target located at the range $OT = OP' - 1000$ yards. From the time of flight to the point P the fuze setting may be determined.

The entering argument (Column 1) for starshell range tables is the *burst range*, i.e., the abscissa of the trajectory for the point having an ordinate of 1500 feet (or 1000 feet in the case of 3" projectiles), rather than the horizontal range of the trajectory. Column 2 gives the angle of departure of the trajectory, Column 3 the corresponding sight-bar range, and Column 4 the time of flight, for the burst range given in Column 1.

If a target at a range of 5000 yards is to be illuminated, and it is desired that the star be released at a point 1000 yards beyond the target and 1500 feet above the surface, i.e., at a point having an abscissa of 6000 yards and an ordinate of 1500 feet, the range table must be entered at range 6000 yards in Column 1 and the corresponding sight-bar range found in Column 3. This sight-bar range will then cause the projectile to travel in a trajectory whose ordinate will be 1500 feet at the burst range of 6000 yards. The fuze setting required to cause a burst at this point is given in Column 5. The variation in the burning rate of the fuze-train due to change of altitude is accounted for in the tabulated fuze settings.

The more recent practice is to prepare the data for illumination fire in the form of graphs rather than in the form of tables. In addition to graphs from which the sight-bar range, time of flight, and fuze setting corresponding to a given burst range may be obtained, graphs are provided also for the determination of corrections to the sight-bar range and fuze setting for variations in initial velocity and atmospheric density.

1820. A peculiar feature of illumination fire is that the sight-bar range may increase while the target distance decreases. The reason for this will be apparent from an examination of Figure 56. If the target were located at T_1 instead of at T ,

the trajectory would have to reach the 1500-foot ordinate at P_1' instead of at P' , and in order to accomplish this the sight-bar range would have to be *increased* from OH to OH_1 . It will be observed that if the target lies beyond the maximum ordinate of the trajectory, an *increase* in target distance will require an *increase* in sight-bar range; but if the target lies short of the maximum ordinate, a *decrease* in target distance will require an *increase* in sight-bar range. It is as a result of this feature that in range tables for illumination fire the sight-bar range (Col. 3) decreases with increases of burst range (Col. 1) up to a certain point, while beyond that point sight-bar range increases with increases of burst range. The fuze setting, however, always increases with increases of burst range.

1821. Ballistic corrections are used in illumination fire chiefly for the purpose of adjusting the relation between sight-bar range and fuze setting for the given conditions of initial velocity and atmosphere, in order that the burst may occur at the required altitude. The nature of illumination fire is such that it is very unlikely that it will be practicable to determine ballistic corrections for other elements, such as wind, firing-ship motion, and target motion. The first stage of illumination fire usually involves a search for the target, which is accomplished by covering wide arcs of the horizon with starshell. When the target has been located, the determination of the sight-bar range and deflection required for efficient illumination is largely a spotting problem. Knowledge of the proper relation between sight-bar range and fuze setting is important, however, in order that the burst points may be maintained at the proper altitude as the sight-bar range is changed.

1822. At some target practices it is desirable to have a preliminary estimate of the starshell deflection. The problem of determining the deflection required to cause the starshell to burst directly in line with the target is essentially the same as that of determining the deflection required to hit the target. However, a point directly in line with the target is usually not the most satisfactory position for the point of burst, for after the star has been released and is suspended in the air, its position with relation to the line of sight is altered by the motion of the target and firing ship, as well as by the wind. Thus if the burst occurs directly in line with the target (as viewed from the firing ship), it will probably soon drift away from this line, especially if there is a strong cross wind or if the target's motion across the line of fire is great. The burst therefore should occur at such a point that the star will drift across the line of sight at approximately the middle of the burning period of the star. The deflection required to give a burst at this point (assuming, of course, that the gun is aimed at the target as usually) can be obtained by combining with the deflection for a burst point directly in line with the target, additional corrections for the effects of true wind and motion of target and firing ship during one-half the burning period of the star. Starshell range tables (or graphs) do not include data as to the effects of wind or motion of gun and target, but approximate values of these effects can be obtained from the regular range table by entering the latter with the time of flight corresponding to the burst point, provided that the form of the starshell does not differ greatly from that of the projectile for which the regular range table has been prepared. The following example illustrates the features outlined above.

Let us suppose that it is desired to determine approximately the starshell deflection for the 5" long-point starshell under the following conditions; the firing ship and target are abeam of each other, port to port, on parallel courses, each steaming at 10 knots; a true wind of 10 knots is blowing from directly astern of

the firing vessel; the target distance is 7000 yards; service charges (3150 f.s.) are to be used. From the graphs for the 5" long-point starshell it is found that the time of flight corresponding to a burst range of $7000 + 1000 = 8000$ yards is 13 seconds (the corresponding fuze setting is 17 seconds, but we are not concerned with the latter in this problem). The 5" long-point starshell is sufficiently similar to the 5" service projectile to permit the use of the regular 5"3150 f.s. range table for determining approximate values of the drift, and of the effects of target and firing-ship motion and of wind.

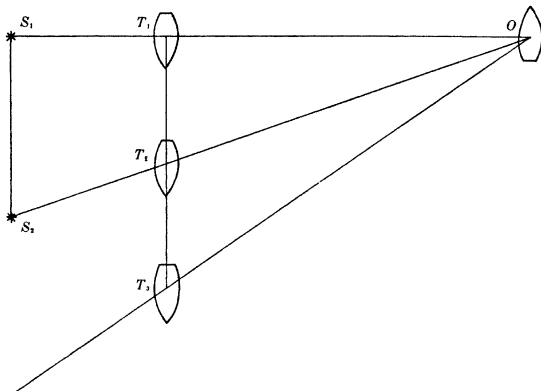


FIGURE 57

We shall first compute the deflection required to give a burst directly in line with the target. Entering the 5"3150 f.s. range table where time of flight 13 seconds appears in Column 4, which occurs at a range of about 8400 yards, we find the following data:

		Error (yds.)
Deflection for a burst point in line with the target	Drift (Col. 6).....	41 R
	Wind (Col. 16).....	27 R
	Firing-ship motion (Col. 17).....	46 R
	Target motion (Col. 18).....	73 R
	Total.....	187 R

Since at the burst range of 8000 yards 1 mil is the equivalent of 8 yards, the total deviation of 187 yards to the right requires a correction of about Left 23 mils (Scale 77).

Let us now examine, in turn, the effects of wind, and of motion of target and firing ship, on the position of the star with respect to the line of sight; and let

us assume that the burning period of the starshell is 40 seconds. In 40 seconds the 10-knot wind will carry the star a distance of 222 yards to the right. In order that the motion imparted to the star by the wind may cause the star to drift across the line of sight at the middle of the burning period, the burst should then occur 111 yards to the left of the line of sight; in other words, the error of the position directly in line with the target is 111 yards *right* due to *wind* alone.

The effect of target motion alone during the burning period is illustrated in Figure 57, in which O and T_1 , respectively, represent the positions of the firing ship and target at the instant at which a burst, S_1 , occurs directly in line with the target. During the burning period of the star the target moves to T_2 , causing the burst at S_1 to be much too far to the right. In order that the *relative* motion of the star with respect to the target may be such as to cause the star to cross the line of sight at the middle of the burning period, the burst should then occur at S_2 , or in line with the position that the target will occupy at the middle of the burning period. The distance T_1T_2 is equal to the travel of the target, at its speed of 10 knots, during one-half the burning period (20 seconds), or it is equal to 111 yards. We have $OT_1 = 7000$ yards, and $T_1S_1 = 1000$ yards. Also

$$\frac{S_1S_2}{111} = \frac{8000}{7000}$$

whence $S_1S_2 = 127$ yards. Thus it is found that the burst should occur 127 yards to the left of the line of sight, and hence the error of the position directly in line with the target is 127 yards *right* due to *target motion* alone.

The effect of firing-ship motion alone during the burning period is illustrated in Figure 58, in which O_1 and T , respectively, represent the positions of the firing ship and target at the instant at which a burst, S_1 , occurs directly in line with the target. During the burning period of the star the firing vessel moves to O_2 , causing the burst at S_1 to be too far to the right. In order that the relative motion of the star with respect to the target may be such as to cause the star to cross the line of sight at the middle of the burning period, the burst should then occur at S_2 , or in line with the position that the firing ship will occupy at the middle of the burning period. We have, in this case, $O_1O_2 = 111$ yards, $O_1T = 7000$ yards, and $TS_1 = 1000$ yards, whence

$$\frac{S_1S_2}{111} = \frac{1000}{7000}$$

and $S_1S_2 = 16$ yards. Thus it is found that the burst should occur 16 yards to the left of the line of sight, and hence the error of the position directly in line with the target is 16 yards *right* due to *firing-ship motion* alone.

The deflection required for a burst point that will cause the star to cross the line of sight at the middle of the burning period may now be found by combining the several errors found above with those previously found. Let us compare the resultant starshell deflection with the deflection that would be required under the stated conditions for fire directed at the target itself. The elements for the target deflection are found from the 5°3150 f.s. range table for the range 7000 yards.

	Error (yds.)	
	Starshell	Target shell
Drift (Col. 6).....	41 R	23 R
Wind (Col. 16).....	27 R	18 R
Firing-ship motion (Col. 17).....	46 R	38 R
Target motion (Col. 18).....	73 R	56 R
Displacement for wind, $\frac{1}{2}$ burning period.....	111 R	—
Displacement for firing-ship motion, $\frac{1}{2}$ burning period..	16 R	—
Displacement for target motion, $\frac{1}{2}$ burning period.....	127 R	—
Total error (yds.).....	441 R	135 R
Correction (mils).....	Left 55	Left 19

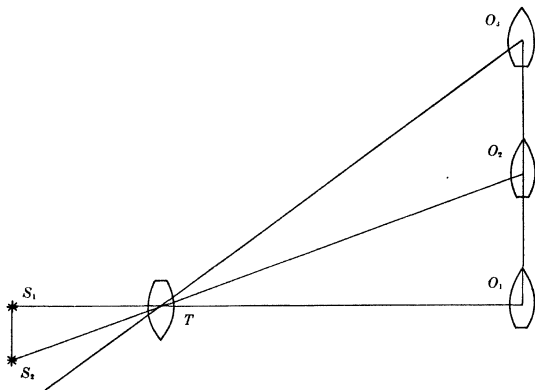


FIGURE 58

The determination of starshell deflection in the manner outlined above is, of course, a highly artificial procedure insofar as its application to a specific case is concerned. But a knowledge of the elements that enter into the deflection for illumination fire will enable the spotter better to anticipate the extraordinary deflections that often occur in this type of fire. Analysis of starshell deflections for typical situations, by the method outlined above, has been put to practical use in the design of devices for the control of deflection in illumination fire.

APPENDIX D

THE EFFECT OF WIND ON THE PRECESSION OF A ROTATING PROJECTILE. THE EFFECT OF CROSS WIND ON THE RANGE OF THE PROJECTILE.

Introductory Note

Although it is a well established fact that cross winds affect the range appreciably at the ranges of Short Range Practice, proving-ground data as to the amount of this effect are totally lacking, and corrections therefor are not given in the range tables. Corrections of an arbitrary nature, designed to account for observed differences between the hitting ballistics of the starboard and port batteries at Short Range Practice, were used perhaps as long as fifteen or more years ago, but it has been only comparatively recently that these differences have been definitely associated with the precessional forces set up by cross winds. However, up to the present time very little has been accomplished toward developing fully the theory governing the action of wind on a rotating projectile, and practically nothing on this subject is to be found in many otherwise complete works on ballistics.

The principal purpose of this appendix is to outline methods for analysis of the effect of cross wind on range and for determination of corrections therefor. Comparison of hitting ballistics for firings to starboard and port at Short Range Practice constitutes, at the present time, practically the sole source of dependable information as to the amount of this effect; the methods outlined herein are accordingly based on such information, and the examples given are confined to short ranges. A brief discussion of the theory governing the action of wind on a rotating projectile is also given, but the purpose of this is merely to give a logical explanation of the observed effect and thus to promote a better understanding of the methods for dealing with it, rather than to provide a basis for theoretical determination of the amount of the effect.

General inadequacy of wind corrections given in range tables 1901. The formulas for the effects of range and cross winds which have been deduced in Chapter 11, and on which Columns 13 and 16 of the range table are based, do not take into account the effects of such winds on the precession of the projectile. Wind obviously operates to alter the pressure which is exerted against the projectile in its flight, and hence operates to alter the projectile's precessional motion and its drift. Complete analysis of the effects of wind on the precessional motion of the projectile is beset with the same difficulties that have already been discussed in connection with the causes of the normal drift (Chapter 9). A further obstacle results from practical difficulties involved in determining experimentally, ballistic effects that may definitely be associated with wind. On the whole, it must be accepted that our knowledge as to the effects of wind on the flight of a rotating projectile is very incomplete, and that the existing wind formulas, which neglect the precessional effects, are in some cases but rough approximations.*

The discussion which follows is limited to a brief examination of the effects of

* Brief mention of the problem involved in calculating the effect of wind on a rotating projectile is made in §5.13, *The Aerodynamics of a Spinning Shell*, by Fowler, Gallop, Lock and Richmond (Philosophical Transactions of the Royal Society of London, Vol. 221, 1921). A practical method for calculating this effect is given in a paper entitled, *The Approximate Motions of Projectiles*, by Commander W. A. Lee, U. S. Navy (Battleships Gunnery Conference, 1932).

both range and cross winds on the precessional motion of the projectile, and is designed principally to give a physical explanation of certain observed results which cannot otherwise be accounted for.

1902. Let us consider first the situation that exists near the beginning of the trajectory of a projectile which is fired into an opposing range wind (i.e., a horizontal wind contrary to the direction of fire). The wind exerts a pressure against the under side of the projectile and tends to cause a clockwise precession of the projectile's axis about the horizontal (assuming that the projectile has right-handed rotation). In any practical case, however, the precessional moment set up by the wind is small in comparison with the precessional moment set up by the air resistance, and the revolution about the horizontal which the wind tends to set up is opposed by the forces tending to keep the axis of the projectile nearly in coincidence with the tangent to the trajectory, which result from the action of the air resistance. Consequently the wind continues to exert itself against the under side of the projectile as long as the latter's axis is pointed above the horizontal, i.e., as long as the projectile is in the ascending branch of the trajectory. The wind pressure, although not great enough to cause complete precessional revolutions of the projectile about the axis of the wind, continuously superimposes upon the motion of the projectile a component tending to drive the latter to the right (since the pressure, being exerted continuously against the under side of the nose, tends to turn the latter to the right). An opposing range wind therefore causes an *increase* in the normal drift in the *ascending* branch of the trajectory. The effect of the wind in thus increasing the drift is greatest near the origin, where the inclination of the projectile's axis with respect to the horizontal wind is greatest, and it vanishes at the summit.

In the descending branch, with an opposing wind, the wind pressure is exerted against the upper side of the projectile. This pressure tends, as before, to cause clockwise precession of the projectile about the horizontal, which is opposed by the forces tending to keep the axis of the projectile nearly in coincidence with the tangent. The situation is therefore the same as in the ascending branch, except that the wind pressure is now exerted continuously against the projectile's upper side, in consequence of which there is superimposed on the latter's motion a component tending to drive it to the left and hence to *decrease* the drift.

The situation in the case of an *opposing* range wind then is that the drift is increased in the ascending branch of the trajectory and decreased in the descending branch. The total *decrease* of drift which occurs in the descending branch exceeds the total *increase* which occurs in the ascending branch, the effect in the descending branch being the greater due to the greater time of flight, greater average inclination, and lesser average velocity of this branch as compared to the ascending branch. Considering that the total effects in each of the two branches are of a relatively small order of magnitude, it is a reasonable assumption that the difference between the effects for the two branches, or the resultant effect for the entire horizontal trajectory, is small enough to be neglected in any practical case.

It may be reasoned similarly that in the case of a range wind which blows *in* the direction of fire, the drift is *decreased* in the *ascending* branch of the trajectory and *increased* in the *descending* branch, and that the resultant effect for an entire horizontal trajectory is a slight *increase* in the drift.

1903. In the case of a horizontal cross wind, the wind pressure is exerted continuously against one side or the other of the projectile throughout the tra-

Effect of
cross wind
on preces-
sion of the
projectile

jectory, and tends to set up a clockwise precession of the projectile's axis about the direction from which the wind is blowing. If unopposed, this precession would cause the projectile to revolve and over end in a vertical circle, the point moving downward when the wind meets the left-hand side of the projectile and upward when the wind meets the right-hand side of the projectile. In any practical case, however, the precessional moment set up by the cross wind is small in comparison with the precessional moment set up by the air resistance, and the revolution which the wind tends to set up is opposed by the forces tending to keep the axis of the projectile nearly in coincidence with the tangent to the trajectory, which result from the action of the air resistance. Consequently the wind exerts itself continuously, throughout the trajectory, against the left-hand side of the projectile in the case of a wind blowing from left to right, and against the right-hand side of the projectile in the case of a wind blowing from right to left.*

The pressure of the cross wind, although not great enough to cause complete precessional revolutions of the projectile about the axis of the wind, continuously superimposes upon the motion of the projectile a component which, in the case of a wind blowing from left to right tends to depress the projectile's nose, and in the case of a wind blowing from right to left tends to elevate the projectile's nose. The action of the cross wind thus results in what may be termed a *vertical drift*, the latter being downward and hence shortening the projectile's flight and range when the cross wind strikes the projectile from the left, and upward and hence sustaining the projectile's flight and increasing its range when the cross wind strikes the projectile from the right.

1904. The conclusions arrived at above therefore are that a *cross wind blowing from left to right decreases the range and a cross wind blowing from right to left increases the range*. It is noteworthy that the effect of a cross wind is of constant direction throughout the trajectory, and also that the wind continues to meet the axis of the projectile practically perpendicularly, and hence with maximum effect, throughout the trajectory. It follows, then, that the change in point of fall of a horizontal trajectory, due to the precessional effect of wind, is much greater in the case of a cross wind than in the case of a range wind. The latter conclusion has been substantiated by practical experience at least to the extent that material changes of range which are clearly attributable to the effects of cross winds have been observed at some target practices, whereas similar evidence as to the effects of range winds on the drift is totally lacking.

1905. Experimental data pertaining to changes in range due to cross winds have been derived principally from analysis of firings at Short Range Practice. This practice lends itself especially well to analysis of the effects of cross winds, for the firings of the starboard and port batteries usually involve large changes in the cross-wind components, without material changes in the many other factors that influence the result. It is a well known fact that at Short Range Practice the hitting ballistics (in range) of the starboard and port batteries, firing under otherwise nearly identical conditions, usually differ materially. The explanation of this rests in the fact that the cross-wind components for the two batteries are usually opposite to each other in direction.

Sources of
data as to
effects of
cross wind
on range

* The terms *right* and *left*, as used hereafter throughout this discussion, are to be considered as signifying directions referred to the direction of fire.

In the case of fire from a moving ship, the projectile is acted upon by an *apparent wind* which is the resultant of the true wind and of the ship's motion (art. 1217). At Short Range Practice ships usually select firing courses which bring the apparent wind a few points on the bow. Under these conditions the direction of the cross wind is from left to right for the starboard battery, and from right to left for the port battery; other things being equal, it is then to be expected that greater sight-bar ranges will be required for firing to starboard than for firing to port, and this is, indeed, found to be the case. In other words, when the apparent wind is from ahead the cross-wind effect requires that the ballistic corrections in range include an additional *up correction* for the starboard battery, and an additional *down correction* for the port battery.

It is to be noted, however, that the above difference between the ballistic corrections in range for the starboard and port batteries depends directly upon the amount and direction of the cross wind for each battery, and that it is associated with the difference in location between the two batteries only insofar as the latter affects the direction of the line of fire with respect to the direction of the apparent wind.* Due to the forward motion of the firing ship, the apparent wind is more often from forward of the beam than from abaft the beam, and it follows that the condition requiring up corrections for the starboard battery and down corrections for the port battery (for cross-wind effect) occurs more often than that requiring the reverse. It must be borne in mind, however, that when the apparent wind blows from a direction abaft the beam, down corrections are required to starboard and up corrections to port.

1906. The following examples illustrate a convenient method of analyzing target-practice results for the purpose of obtaining values of the correction in range due to the effect of cross wind.

Analysis of
cross-wind effect
on range from
S.R.B.P. firing

I. *Given:* The post-firing computed *gun ballistics* in range (Sheet 10) for the Secondary Battery groups of a battleship, for Short Range Practice, were as follows: starboard battery, *Up* 117 yards; port battery, *Up* 128 yards. The *hitting navigational-range ballistics* for the same groups were: starboard battery, *Up* 207 yards; port battery, *Up* 90 yards. The cross winds (i.e., components of apparent wind perpendicular to the line of fire) were as follows for these groups: starboard battery, 14 knots *from the left*; port battery, 26 knots *from the right*.

Find: The *correction* in range (in yards per knot) due to cross wind, and the residual arbitrary ballistic, corresponding to the above data.

* The fact that Short Range Practice is usually fired with the apparent wind from ahead, under which condition a projectile fired to port actually travels farther than one fired to starboard (other things being equal), has led to a popular misconception that this is *always* the case. Evidence of this is to be seen in the term "port-side effect" which is commonly used in referring to the difference between the hitting ballistics of the two batteries. This term is misleading in that it associates the difference between the two batteries with their locations on the ship, whereas, in fact, the difference depends entirely on the direction of the cross wind, as noted above.

The term "Magnus Effect" is also commonly used in referring to the change of range caused by cross wind. However, the "Magnus Effect" due to cross wind causes vertical drift upward, and hence increase of range, for a cross wind blowing from left to right,—and vertical drift downward, and hence decrease of range, for a cross wind blowing from right to left (see art. 904). The effects actually observed in practice occur in the opposite directions, and hence cannot be attributed to "Magnus Effect."

We have, to begin with, the following differences between the hitting (or actual) ballistics and the computed (or predicted) ballistics for the two batteries:

	Port battery	Starboard battery
Hitting ballistic.....	+ 90 yds.....	+207 yds.
Computed ballistic.....	+128 yds.....	+117 yds.
Difference.....	- 38 yds.....	+ 90 yds.
Cross wind*.....	- 26 kts.....	+ 14 kts.

It is at once apparent that these entire differences are not proportional to their respective cross-wind components, and that the differences must include something besides the changes due to cross-wind effect. Let us assume, then, that the difference for each battery includes, (1) a correction in range due to cross-wind effect, proportional in amount to the component of cross wind for that battery, and (2) a residual arbitrary correction which is common to both batteries. Let us denote by x (in yards) the *correction* in range due to 1 *knot* of cross wind; according to the conclusions stated in articles 1904 and 1905, x will be negative for wind from the right and positive for wind from the left. Let us denote by y (in yards) the residual arbitrary ballistic (which is, by assumption, the same for both batteries). We may now set up the following relations:

$$\text{Starboard battery} \dots\dots\dots y + 14x = 90$$

$$\text{Port battery} \dots\dots\dots y - 26x = -38$$

whence we find, $x = 3.2$ yards per knot, and $y = 45$ yards.

In other words, we have found that the *correction* in range for the effect of cross wind is 3.2 yards per knot (of cross wind), and that the residual arbitrary ballistic is *Up* 45 yards. The data of this example are fairly typical for the firing of a 5"2300 f.s. gun (reduced charge) at Short Range Practice, at ranges of 1600 to 1700 yards.

The following is an example of the analysis for a case in which the cross wind is in the same direction, with respect to the line of fire, for both batteries; this condition may arise if the firing vessel is obliged to change the base course between the firings of the two batteries.

II. *Given:* The post-firing computed *gun ballistics* in range (Sheet 10) for the Secondary Battery groups of a battleship, for Short Range Practice, were as follows: starboard battery, *Up* 96 yards; port battery *Up* 82 yards. The *hitting navigational-range ballistics* for the same groups were: starboard battery, *Up* 173 yards; port battery, *Up* 91 yards. The cross winds were as follows for these groups: starboard battery, 32 knots *from the left*; port battery, 12 knots *from the left*.

Find: The *correction* in range (in yards per knot) due to cross wind, and the residual arbitrary ballistic, corresponding to the above data.

Proceeding just as in the first example, we have:

	Port battery	Starboard battery
Hitting ballistic.....	+91 yds.....	+173 yds.
Computed ballistic.....	+82 yds.....	+ 96 yds.
Difference.....	+ 9 yds.....	+ 77 yds.
Cross wind*.....	+12 kts.....	+ 32 kts.

* The signs used with the cross-wind components here agree with the signs of the corresponding range corrections.

Starboard battery.....	$y + 32x = 77$
Port battery.....	$y + 12x = 9$

whence we find $x = 3.4$ yards per knot, and $y = -32$ yards. In other words, the correction in range for the effect of cross wind is 3.4 yards per knot (of cross wind), and the residual arbitrary ballistic is *Down* 32 yards.

1907. Analysis of Short Range Practice results has shown beyond doubt that at the ranges of this practice (1600–2100 yards) cross wind affects the range materially, especially in the case of the smaller calibers. Values of from 3.0 to 3.5 yards per knot are generally found in the case of the 5"/51 2300 f.s. gun.* Other typical values are: about 2.5 yards per knot for the 5"/25 2200 f.s. gun; about 1.5 yards per knot for the 6"/53 2300 f.s. gun; about 2 yards per knot for the 4"/50 2900 f.s. gun; about 1 yard per knot for major calibers at their target-practice velocities (14" and 16").

It is noteworthy that at the ranges of Short Range Practice the effect on *range* of wind in the line of fire is practically negligible in comparison with the effect on *range* of wind *across* the line of fire. For example, Column 13 of the 5"/51 2300 f.s. range table shows that at 1700 yards a 10-knot wind *in* the line of fire causes a change of range of 3 yards; analysis of firings at this range shows that a 10-knot wind *across* the line of fire causes a change of range of about 30 or 35 yards, or a change at least ten times as great as that due to wind *in* the line of fire. The ratio in the cases of other guns is also of the order of about ten to one. It should be understood, however, that the situation here described exists particularly at very short ranges. The situation for long ranges will be discussed later.

Determination and application of range corrections for wind at S.R.P.

1908. The following examples illustrate the method of determining and applying corrections to the range for the components of wind in and across the line of fire.

I. *Given*: The Secondary Battery of a battleship is to fire its 5"/51 guns, at Short Range Practice, at the reduced velocity of 2300 f.s. The ballistic corrections in range have been computed in advance of the practice for all items except apparent wind, and the total ballistic (less wind) for each battery has been found to be as follows: starboard battery, *Up* 122 yards; port battery, *Up* 116 yards. From analysis of previous firings of this battery it has been determined that a correction in range of 3.2 yards per knot is required for the effect of cross wind. The value in Column 13 of the 5"2300 f.s. range table,† at 1700 yards, is 3 yards. Just prior to the firing of each battery the apparent wind was observed to be as follows: starboard battery, 30 knots from 15° relative; port battery, 27 knots from 340° relative. The mean relative bearing of the target was expected to be practically 90° for the starboard side, and 270° for the port side.

Find: The total ballistic corrections in range (including wind) for the two batteries.

* This applies to the 5"/51 guns with uniform-twist rifling of 1 turn in 35 calibers; values around 4 yards per knot were found for the 5"/51 gun with increasing-twist rifling (final twist 1 in 25) which was formerly in use.

† Note that this is not the 5" range table which is given in *Range and Ballistic Tables*, 1955.

Let us first resolve the wind, for each battery, into its components *in* and *across* the line of fire (ref. arts. 1215-1216). We have,

	<u>Port battery</u>	<u>Starboard battery</u>
Wind in line of fire	$27 \cos 70^\circ = 9 \text{ kts. (against)}$	$30 \cos 75^\circ = 8 \text{ kts. (against)}$
Wind across line of fire	$27 \sin 70^\circ = 25 \text{ kts. (from right)}$	$30 \sin 75^\circ = 29 \text{ kts. (from left)}$

The *corrections* for the components *in* the line of fire are, using the given value from Column 13,

$$\begin{aligned} \text{Starboard battery} & \dots\dots\dots \frac{8}{10} \times 3 = +2 \text{ yards} \\ \text{Port battery} & \dots\dots\dots \frac{9}{10} \times 3 = +3 \text{ yards} \end{aligned}$$

The *corrections* for the components *across* the line of fire are, using the given value of 3.2 yards per knot,

$$\begin{aligned} \text{Starboard battery} & \dots\dots\dots (+) 29 \times 3.2 = +93 \text{ yards} \\ \text{Port battery} & \dots\dots\dots (-) 25 \times 3.2 = -80 \text{ yards} \end{aligned}$$

The required total corrections then are,

<u>Corrections</u>	<u>Port battery</u>	<u>Starboard battery</u>
Computed (less wind).....	+116 yds.....	+122 yds.
Wind in line of fire.....	+ 3 yds.....	+ 2 yds.
Wind across line of fire.....	- 80 yds.....	+ 93 yds.
Total.....	+ 39 yds.....	+217 yds.

II. *Given:* The same data as for example I of this article, except that the apparent wind observed just prior to the firing of each battery was as follows: starboard battery, 12 knots from 150° relative; port battery, 30 knots from 300° relative.

Find: The total ballistic corrections in range (including wind) for the two batteries.

Proceeding just as in the first example, we have,

	<u>Port battery</u>	<u>Starboard battery</u>
Wind in line of fire	$30 \cos 30^\circ = 26 \text{ kts. (against)}$	$12 \cos 60^\circ = 6 \text{ kts. (against)}$
Wind across line of fire	$30 \sin 30^\circ = 15 \text{ kts. (from right)}$	$12 \sin 60^\circ = 10 \text{ kts. (from right)}$

Corrections for components *in* line of fire,

$$\begin{aligned} \text{Starboard battery} & \dots\dots\dots \frac{6}{10} \times 3 = +2 \text{ yards} \\ \text{Port battery} & \dots\dots\dots \frac{26}{10} \times 3 = +8 \text{ yards} \end{aligned}$$

Corrections for components across line of fire,

Starboard battery.....($-$) $10 \times 3.2 = -32$ yards

Port battery.....($-$) $15 \times 3.2 = -48$ yards

Required total corrections,

Corrections	Port battery	Starboard battery
Computed (less wind).....	+116 yds.....	+122 yds.
Wind in line of fire.....	+ 8 yds.....	+ 2 yds.
Wind across line of fire.....	- 48 yds.....	- 32 yds.
Total.....	+ 76 yds.....	+ 92 yds.

1909. Analysis of cross-wind effects for long ranges, from target-practice results, is rarely practicable, for the conditions attending long-range practices are generally such as to render uncertain a number of the factors upon which the ballistic corrections depend. Considering the degree of approximation with which aloft winds are generally known, and with which the ballistic wind is determined therefrom, it is difficult, in the first place, to discriminate between errors which may be due to the use of incorrect values of the wind itself, and those which may be due to incorrect values as to the effect of the wind. Another fundamental difficulty lies in the uncertainty which usually exists in the measurement of the actual ranges obtained. Analysis of cross-wind effects depends upon accurate knowledge of hitting ballistics under materially varying conditions of cross wind; this, in turn, involves accurate knowledge of hitting gun ranges and navigational ranges for different legs, or phases, of a practice. It is doubtful whether the determination of these values for long-range practices, even by careful post-firing analysis, is ever accomplished with sufficient accuracy to yield dependable values of the cross-wind effect at long ranges.

Such information as has been obtained from analysis of long-range target practices indicates that at such ranges the cross-wind effect on range is relatively small, or, at least, too small to be clearly identified, under the limitations of analysis noted above, among the ordinary errors attending long-range fire. Although this finding appears rather surprising, considering the relatively great effect noted at short ranges, it is, in fact, substantiated by the theory governing the action of the projectile under the influence of cross wind. As noted in article 1903, cross wind causes a vertical drift of the projectile. Since the direction of the cross wind with respect to the direction of the projectile remains practically constant throughout the trajectory, the vertical drift increases approximately in proportion to the time of flight.* But the change of range due to a given amount of vertical drift is equal, approximately, to the vertical drift multiplied by the cotangent of the angle of fall. As the range increases, the cotangent of the angle of fall decreases more rapidly than the time of flight increases, whence it follows that, even although the vertical drift increases with range, the change of range due thereto decreases with range. In the case of the 5"2300 f.s. gun it is likely that the cross-wind effect on range at 10,000 yards does not exceed about 2 yards per knot, as compared to 3 or 3.5 yards per knot at 1700 yards. The effect for long ranges may

* Change in stability of the projectile as it loses velocity, while the cross wind remains constant, alters the rate of vertical drift somewhat. Commander Lee, in his theoretical analysis of the cross-wind effect for the 5" 2300 f.s. and 14" 2000 f.s. guns (see footnote on page 290) finds that the vertical drift in these cases increases somewhat more rapidly than the time of flight, but that the resultant changes of range nevertheless decrease with range.

therefore be regarded as of relatively small importance, in comparison with range-finding errors, errors in measurement of winds and densities, etc., at long ranges.

EXERCISES

I. Given: The post-firing computed gun ballistics in range, hitting navigational-range ballistics, and observed apparent winds, for the starboard and port batteries at Short Range Practice; the mean relative bearing of target was 90° for the starboard battery, and 270° for the port battery.

Find: The correction in range (in yards per knot) due to cross wind, and the residual arbitrary ballistic.

	Given									Answers	
Gun	Port battery				Starboard battery				Cross-wind effect	Residual arbitrary ballistic	
	Computed gun ballistic	Hitting ballistic	Wind		Computed gun ballistic	Hitting ballistic	Wind				
			Velocity	Rel. brg.			Velocity	Rel. brg.			
	yds.	yds.	kts.	deg.	yds.	yds.	kts.	deg.	yds./kt.	yds.	
A	4''2900 f.s.	Up 80	Up 49	28	330	Up 88	Up 172	30	20	2.2	Up 22
B	5''2200 f.s.	Up 65	Down 3	24	350	Up 57	Up 98	21	30	2.6	Down 6
C	5''2300 f.s.	Up 85	Up 177	8	210	Up 75	Up 235	28	10	3.2	Up 70
D	6''2300 f.s.	Up 112	Up 74	33	340	Up 120	Up 125	6	160	1.7	Up 15

II. Given: The pre-firing ballistic corrections in range (less wind), correction in range due to cross wind (in yards per knot), and the apparent winds observed just prior to firing, for the starboard and port batteries at Short Range Practice; the mean relative bearing of target for the starboard battery was expected to be 90° , and for the port battery 270° .

Find: The total ballistic corrections in range (including wind) for the two batteries.

Given										Answers	
Gun	Cross-wind effect on range	Range table, Col. 13*	Port battery				Starboard battery			Total ballistic corrections in range	
			Computed ballistic less wind	Wind		Computed ballistic less wind	Wind		Port	Stbd.	
				Velocity	Rel. brg.		Velocity	Rel. brg.			
	yds./kt.	yds.	yds.	kts.	deg.	yds.	kts.	deg.	yds.	yds.	
A	4''2900 f.s.	2.0	2	Up 90	33	340	Up 104	6	160	Up 30	Up 92
B	5''2200 f.s.	2.5	3	Up 85	8	210	Up 80	28	10	Up 104	Up 152
C	5''2300 f.s.	3.2	3	Up 65	24	350	Up 68	21	30	Down 11	Up 129
D	6''2300 f.s.	1.5	2	Up 70	28	330	Up 88	30	20	Up 37	Up 132

* The values given are for 10-knot components.

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